CDS 101/110: Lecture 2.3
Stability (Continue), MATLAB

7 October 2016

Goals:

- Review stability of linear systems
- A little more detail on Lyapunov Functions
- Start of Tutorial on MATLAB functions for control system analysis

Reading:

- Åström and Murray, *Feedback Systems* 2e, Sections 5.1-5.4

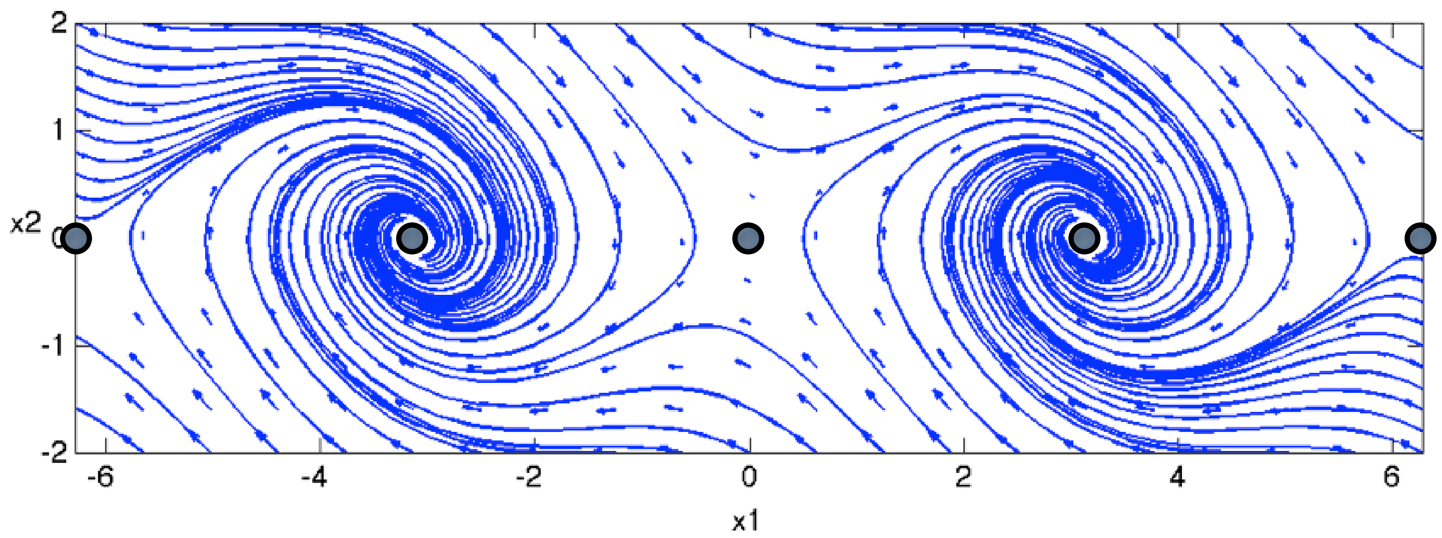
Equilibrium Points

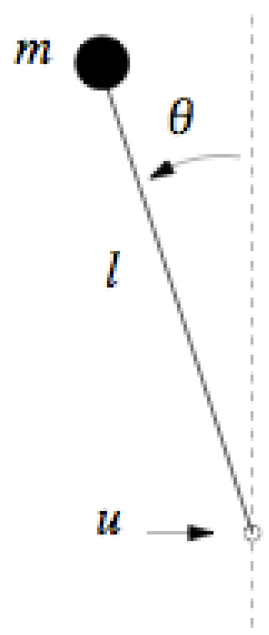
The ***equilibria*** of system $\dot{x} = f(x)$ are the points x_e such that $f(x_e) = 0$.

They represent stationary conditions for the dynamics

- Nonlinear systems may have *multiple* equilibria
- Linear systems have *only one* equilibria (unless characteristic equation has zero eigenvalues)

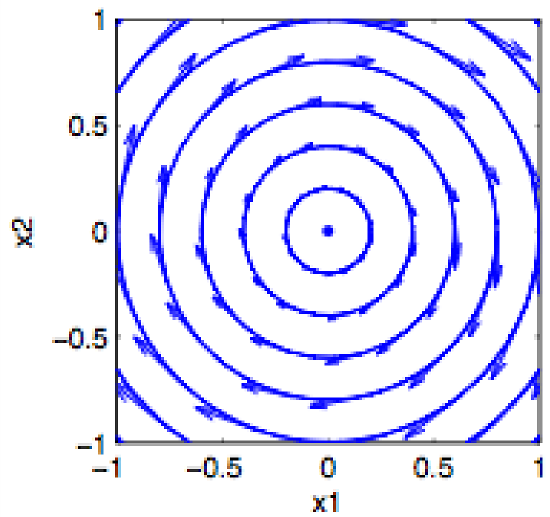
$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix} \quad \Rightarrow \quad x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$





An Equilibrium Point is:

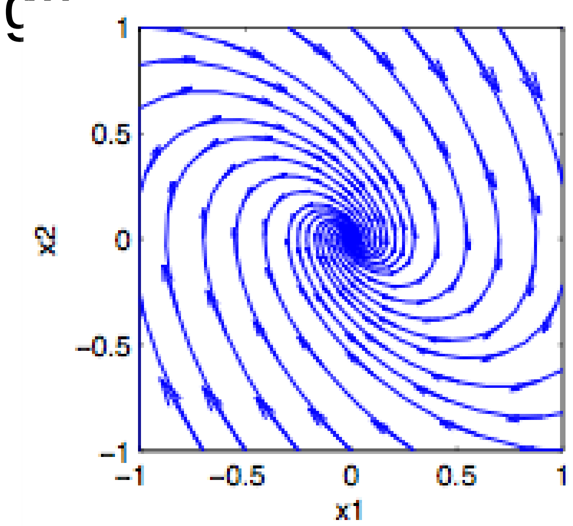
Stable if initial conditions that start near the equilibrium point, stay near the equilibrium point



- Also called "stable in the sense of Lyapunov"
- Technical:
  - For all $\varepsilon > 0$, there exists $\delta > 0 \; s.t.$

$$\left|\left|x(0) - x_e\right|\right| < \delta \quad \rightarrow \quad \left|\left|x(t) - x_e\right|\right| < \varepsilon \quad \forall t \geq 0$$

Asymptotically stable if all initial conditions near equilibrium converge to the equilibrium



- Stable + converging
- Technical:
  - x_e is *locally attractive* if $\exists\delta > 0$ s.t. $\left|\left|\mathrm{x}(0) - \mathrm{x_e}\right|\right| < \delta$ and $\lim_{t\to\infty} x(t) = x_e$
  - x_e is *locally asymptotically stable* if it is locally stable & locally attractive

Exponentially stable if it is asymptotically stable and $\exists\; \alpha, \beta > 0 \; s.t.$

$$\left|\left|\mathrm{x}(t) - \mathrm{x_e}\right|\right| \leq \alpha \left|\left|\mathrm{x}(0) - \mathrm{x_e}\right|\right| \mathrm{e}^{-\beta t} \qquad t \geq 0$$

Linear Systems

Recall: Linearity of Functions $f: \mathbb{R}^n \to \mathbb{R}^n$

- Addition: $f(x + y) = f(x) + f(y)$
- Scaling: $f(\alpha x) = \alpha f(x)$
- Zero at the Origin: $f(0) = 0$

$$\left.\begin{array}{l} f(x+y) = f(x) + f(y) \\ f(\alpha x) = \alpha f(x) \\ f(0) = 0 \end{array}\right\} \begin{array}{l} f(\alpha x + \beta y) \\ \quad = \alpha f(x) + \beta f(y) \end{array}$$

Linear *System:* $S: u(t) \to x(t)$

- If $S: u_1(t) \to x_1(t)$; $S: u_2(t) \to x_2(t)$
  - $\alpha x_1(t) + \beta x_2(t) = S\{\alpha u_1(t) + \beta u_2(t)\}$

Linear *Control System:*

- $\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$
- $y(t) = C(t)\, x(t) + D(t)\, u(t)$

$x(t)$ is system "state";

$u(t)$ are control inputs

$y(t)$ is the system output, (what is observed)

Linear Time Invariant Systems

Linear Time Invariant (LTI) System:

- Given that input $u(t)$ leads to output $y(t)$, if input $u(t+T)$ leads to output $y(t$

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$\begin{aligned} \dot{x} &= Ax + B\cancel{u}^{\,0} \\ y &= Cx + D\cancel{u}^{\,0} \end{aligned} \longrightarrow x(t) = e^{At}x_0 \longrightarrow y(t) = Ce^{At}x_0$$

$$\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$$

Stability of Linear Systems

Coordinate (similarity) Transform: $z = Tx$, where T is invertible

- $\dot{z} = T\dot{x} = TAx = TAT^{-1}z$
- If system has equilibrium $\mathrm{x} = 0$, then $\mathrm{z} = 0$ is also an equilibrium
- If system is stable in x-coordinates, it is stable in z-coordinates
- If T is invertible, $e^{TAT^{-1}} = Te^{A}\mathrm{T}^{-1}$

Diagonalized case: Eigenvectors are unique: *A* is diagonalizable (real, distinct eigenvalues)

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x \quad \Rightarrow \quad x(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \mathrm{O} & \\ 0 & & e^{\lambda_n t} \end{bmatrix} x_0$$

- Stable if $\lambda_i \leq 0$
- Asy stable if $\lambda_i < 0$
- Unstable if $\lambda_i > 0$

Block diagonal case (distinct complex eigenvalues)

$$\frac{dx}{dt} = \begin{bmatrix} \sigma_1 & \omega_1 & & 0 & 0 \\ -\omega_1 & \sigma_1 & & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & & \sigma_m & \omega_m \\ 0 & 0 & & -\omega_m & \sigma_m \end{bmatrix} x$$

$$x_{2j-1}(t) = e^{\sigma_j t}\left(x_i(0)\cos\omega_j t + x_{i+1}(0)\sin\omega_j t\right)$$
$$x_{2j}(t) = e^{\sigma_j t}\left(x_i(0)\sin\omega_j t - x_{i+1}(0)\cos\omega_j t\right)$$

- System is asy stable if $Re(\lambda_i) = \sigma_i < 0$

Stability of Linear Systems: Jordan Form

Most general case is ***Jordan Form.***

- For matrix A there exists invertible T such that $\mathrm{A} = T^{-1}JT$,

$$e^{At} = e^{T^{-1}JT} = T^{-1}e^{J}T = T^{-1}\begin{bmatrix} e^{J_1 t} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{J_k t} \end{bmatrix} T, \qquad J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

where J_i is a ***Jordan Block.*** $\boldsymbol{\lambda_i}$ is the i^th^ eigenvalue of A, m_i is the size of J_i

Remarks:

- Jordan Block decomposition is ***unique*** up to permutation of blocks
- Can have multiple blocks with same eigenvalues
- Solutions with Jordan blocks

$$e^{J_k t} = e^{\lambda_k t}\begin{bmatrix} 1 & t & t^2/2 & \cdots & t^{m-1}/m! \\ 0 & 1 & t & \cdots & \\ 0 & 0 & 1 & t & t^2/2 \\ \vdots & \vdots & \vdots & \ddots & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- If $\mathrm{Re}(\lambda_i) < 0$, $e^{J_i t} z_0 \to 0$
- If $\mathrm{Re}(\lambda_i) > 0$, $e^{J_i t} z_0 \to \infty$
- If $\mathrm{Re}(\lambda_i) > 0$, need more analysis

Stability of Linear Systems

Theorem:

- A linear system $\dot{x} = Ax$ is *asymptotically stable* if and only if $Re(\lambda_i) = \sigma_i < 0$ for all eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of A

Proof:

- Write $z = Tx$ where $T^{-1}AT$ is in Jordan Form
- Solution to o.d.e. is then sum of terms having the form $t^k e^{\lambda t}$
- If $\mathrm{Re}(\lambda) < 0$, then $t^k e^{\lambda t} \rightarrow 0$ as t $\rightarrow \infty$

What about $Re(\lambda_i) = \sigma_i = 0$?

- Stability depends upon Jordan structure.
- If $Re(\lambda_i) = 0$ and $m_i > 1$

$$e^{J_i t} z_0 = \begin{bmatrix} 1 & t & \cdots & t^{m-1}/m! \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & t \\ 0 & \cdots & \cdots & 1 \end{bmatrix} z_0 \quad \rightarrow \infty \qquad as\ t \rightarrow \infty$$

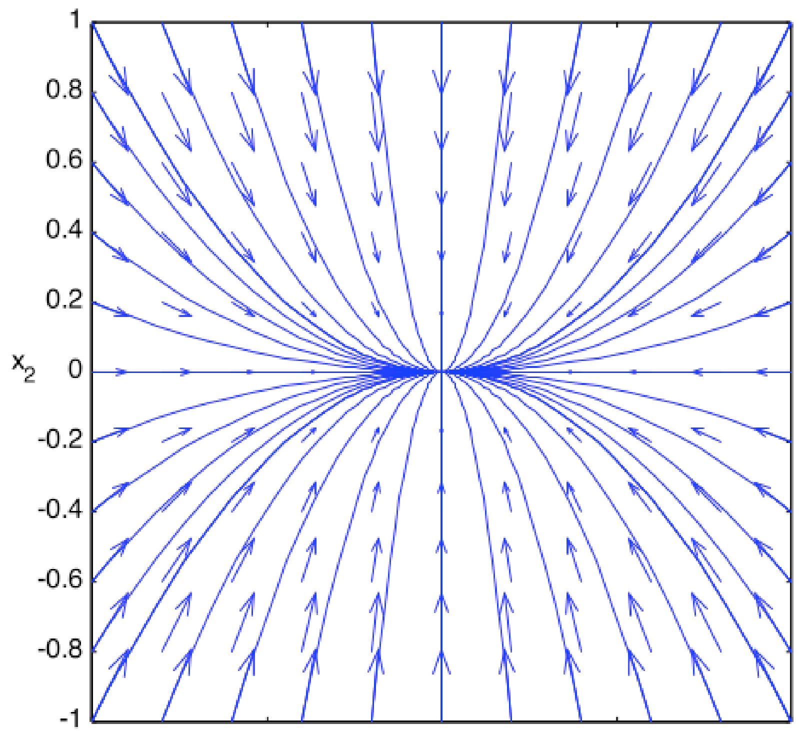
- If $\lambda_i = \pm i$, system is *stable* (but not asymp. Stable)

$$J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad \begin{bmatrix} z_i \\ z_{i+1} \end{bmatrix} = \begin{bmatrix} \mathrm{c}_1 \cos(t) + c_2 \sin(t) \\ -c_3 \sin(t) + c_4 \cos(t) \end{bmatrix}$$

Eigenstructure of Linear Systems

Real e-values

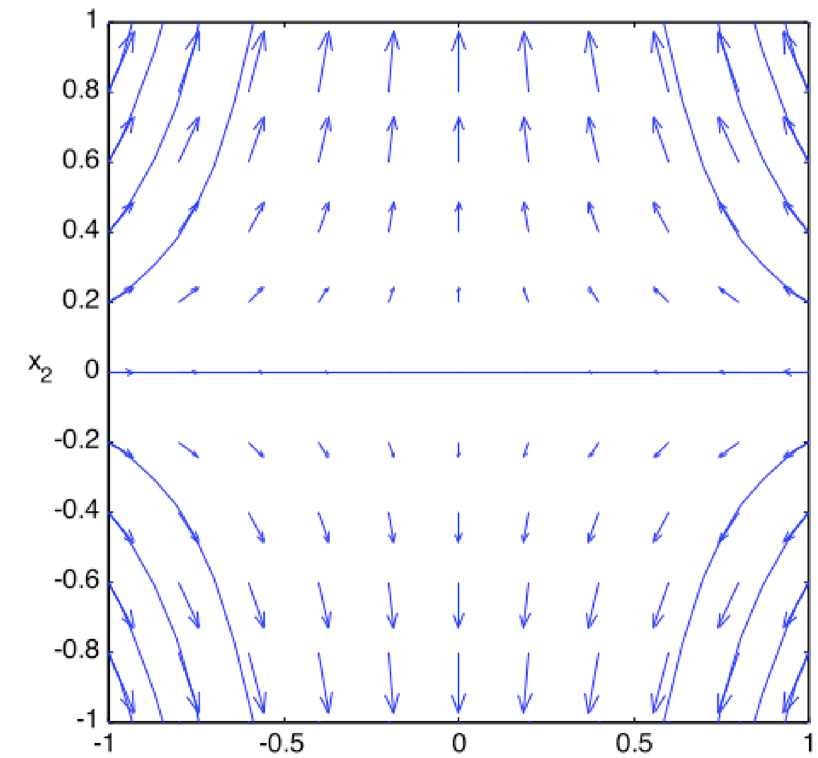
$Re(\lambda_i) < 0$



Real e-values

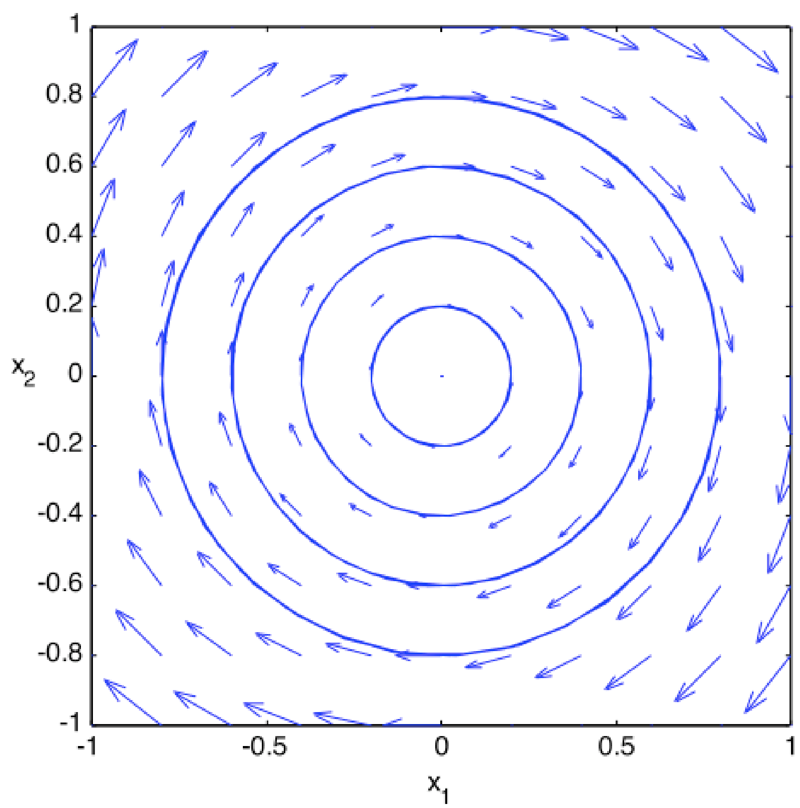
$Re(\lambda_i) < 0$

$Re(\lambda_i) > 0$



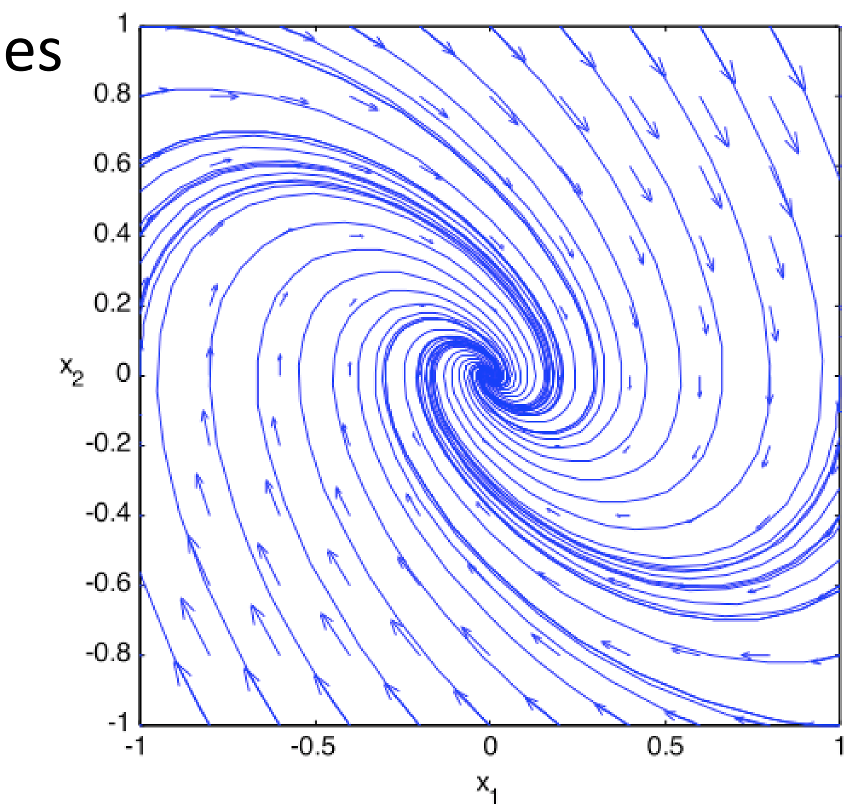
Complex e-values

$Re(\lambda_i) = 0$



Complex e-values

$Re(\lambda_i) < 0$



Linearization Around an Equilibrium Point

$$\dot{x} = f(x,u) \qquad \longrightarrow \qquad \dot{z} = Az + Bv$$
$$y = h(x,u) \qquad\qquad\qquad w = Cz + Dv$$

"Linearize" around $x = x_e$

$$f(x_e, u_e) = 0 \qquad y_e = h(x_e, u_e)$$

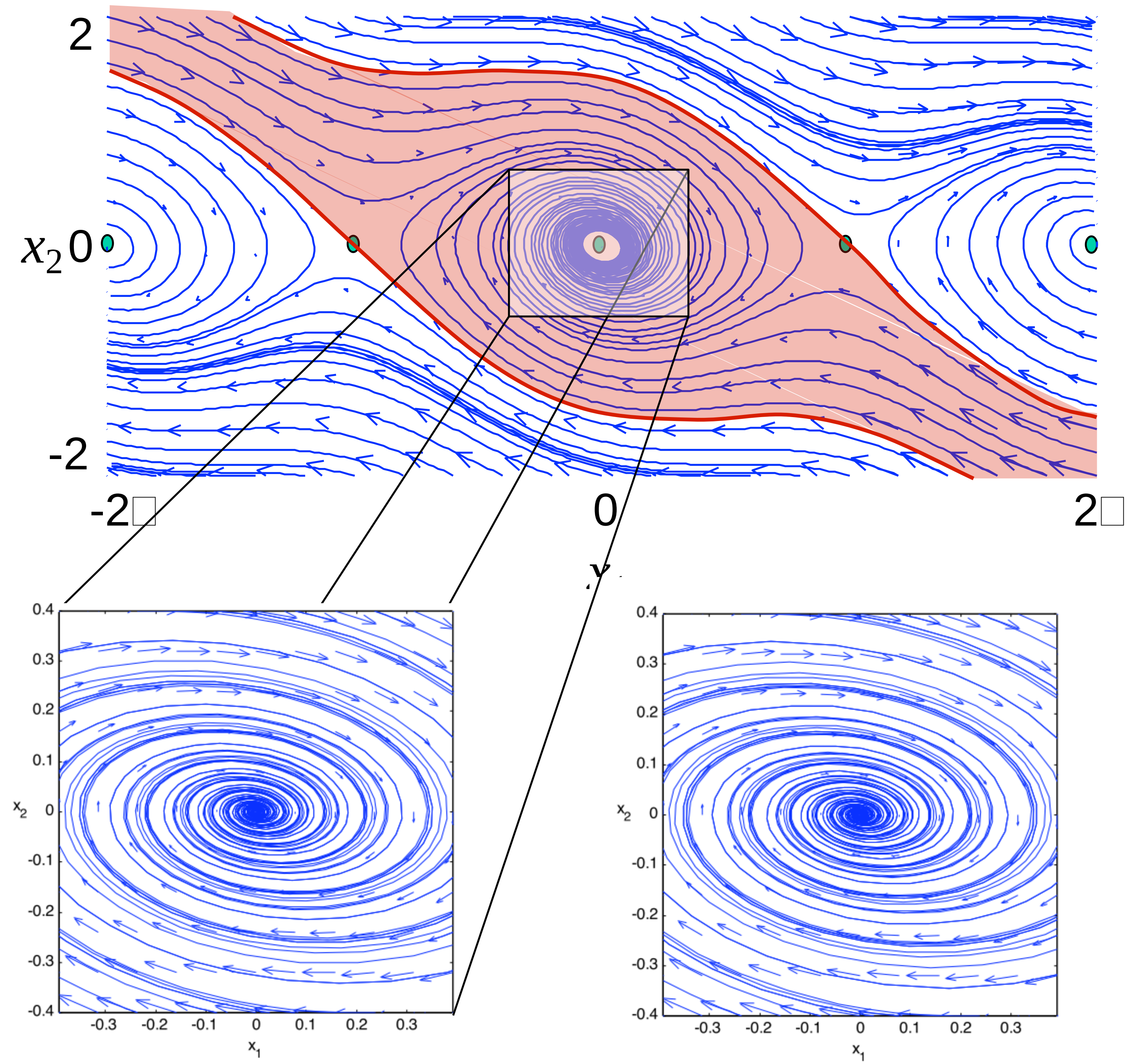
$$z = x - x_e \qquad v = u - u_e \qquad w = y - y_e$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e, u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e, u_e)}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e, u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e, u_e)}$$

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point



Full nonlinear model

Linear model (honest!)

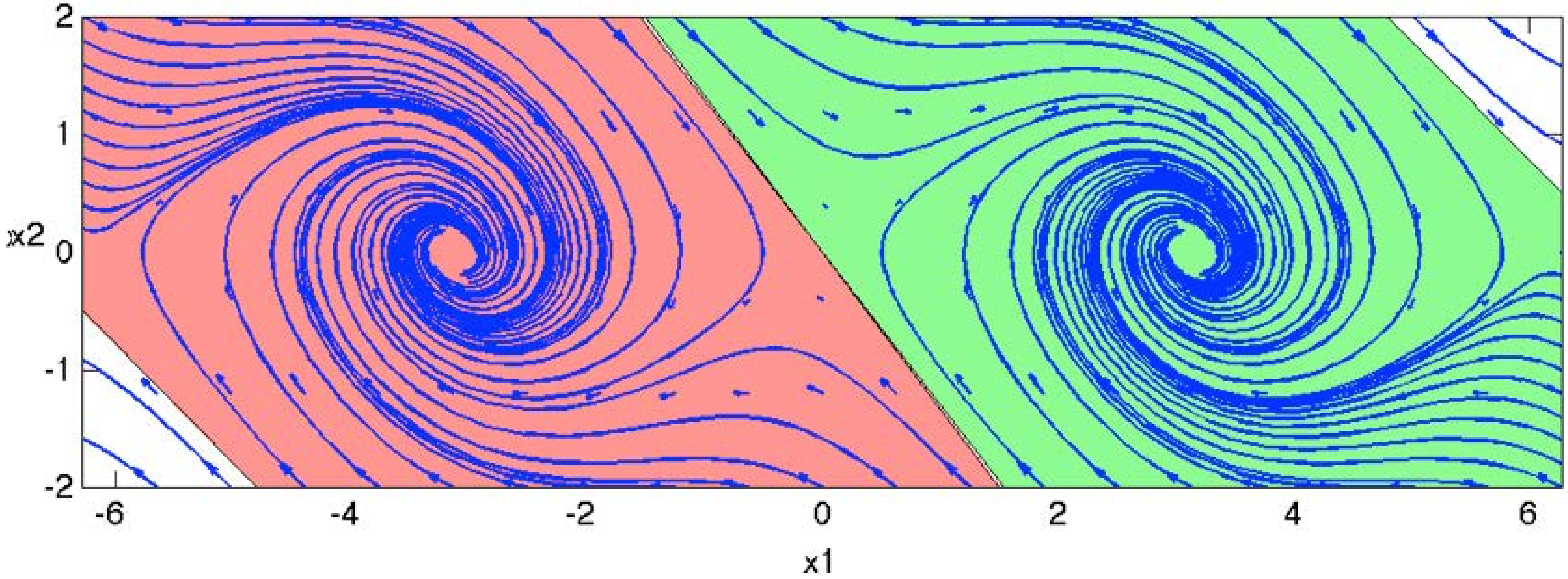
Local versus Global Behavior

Stability is a *local* concept

- Equilibrium points define the local behavior of the dynamical system
- Single dynamical system can have stable *and* unstable equilibrium points

Region of attraction

- Set of initial conditions that converge to a given equilibrium point



Reasoning about Stability using Lyapunov Functions

Basic idea: capture system behavior by tracking its "energy"

- Find a single function that captures distance of system from equilibrium
- Try to reason about the long term behavior of all solutions, without explicit solution!

Technical: A function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a Lyapunov function for $\dot{x} = f(x)$

- $V(0) = 0$, $\quad V(x) > 0 \ \forall x \epsilon B_r \backslash \{0\}$ for B_r a neighborhood of 0
- Note that coordinates can be chosen so that $\mathrm{x_e} = 0$
- $\dot{V}(x) = \frac{d}{dt} V\big(x(t)\big) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x} \cdot f(x)$
- The equilibrium is *locally stable* if $\dot{V}(x) \leq 0 \quad \forall x \epsilon B_r \backslash \{0\}$
- The equilibrium is *locally asymptotically stable if* $\dot{V}(x) < 0 \quad \forall x \epsilon B_r \backslash \{0\}$
- *Artstein's Thm:* if system is stable, a Lyapunov function exists.

Linear Systems: $\dot{x} = Ax$

- Consider $V(x) = x^T P x$, where $P^T = P > 0$ (positive definite)
- $\dot{V}(x) = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = x^T\big(A^T P + PA\big)x = -x^T Q x$
- Lyapunov Equation: $A^T P + PA = -Q$
- Can show that solution P exists if A has eigenvalues in left-half plane.

Reasoning about Stability using Lyapunov Functions

Lasalle's Invariance Principle (Barbashin-Krasvoskii-Lasalle)

- Gives a way to show asymp. Stability when $\dot{V}(x) \leq 0$
- Only for time-invariant or periodic systems

Technical:

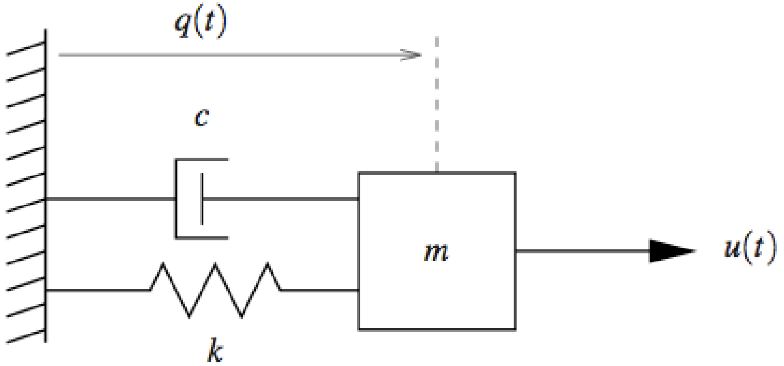
- ω-limit set of a trajectory $x(t, x_0)$ is the set of points p such that $x(t, x_0) \rightarrow p$ as t→ ∞ for $\dot{x} = f(x)$.
- A set M is said to be *invariant* if for all $x_0 \in M$, $x(t, x_0) \in M$ for all $t \geq 0$
- **Theorem (5.4, page 5-25):** Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_r = \{x \,\epsilon \mathbb{R}^n \mid V(x) \leq r\}$, Define

$$S = \{x \,\epsilon \mathbb{R}^n \mid \dot{V}(x) = 0\}.$$

as t→ ∞, the trajectory tends to the largest invariant set in S. If S contains no invariant set except other than x=0, then x=0 is asymptotically stable.

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 \qquad \dot{V} = -cx_2^2.$$

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix} \qquad \begin{aligned} x_1 &= q \\ x_2 &= \dot{q} \end{aligned}$$



Example #2: Predator Prey (ODE version)

Continuous time (ODE) version of predator prey dynamics:

$$\frac{dH}{dt} = rH\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H} \qquad H \geq 0$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL \qquad L \geq 0.$$

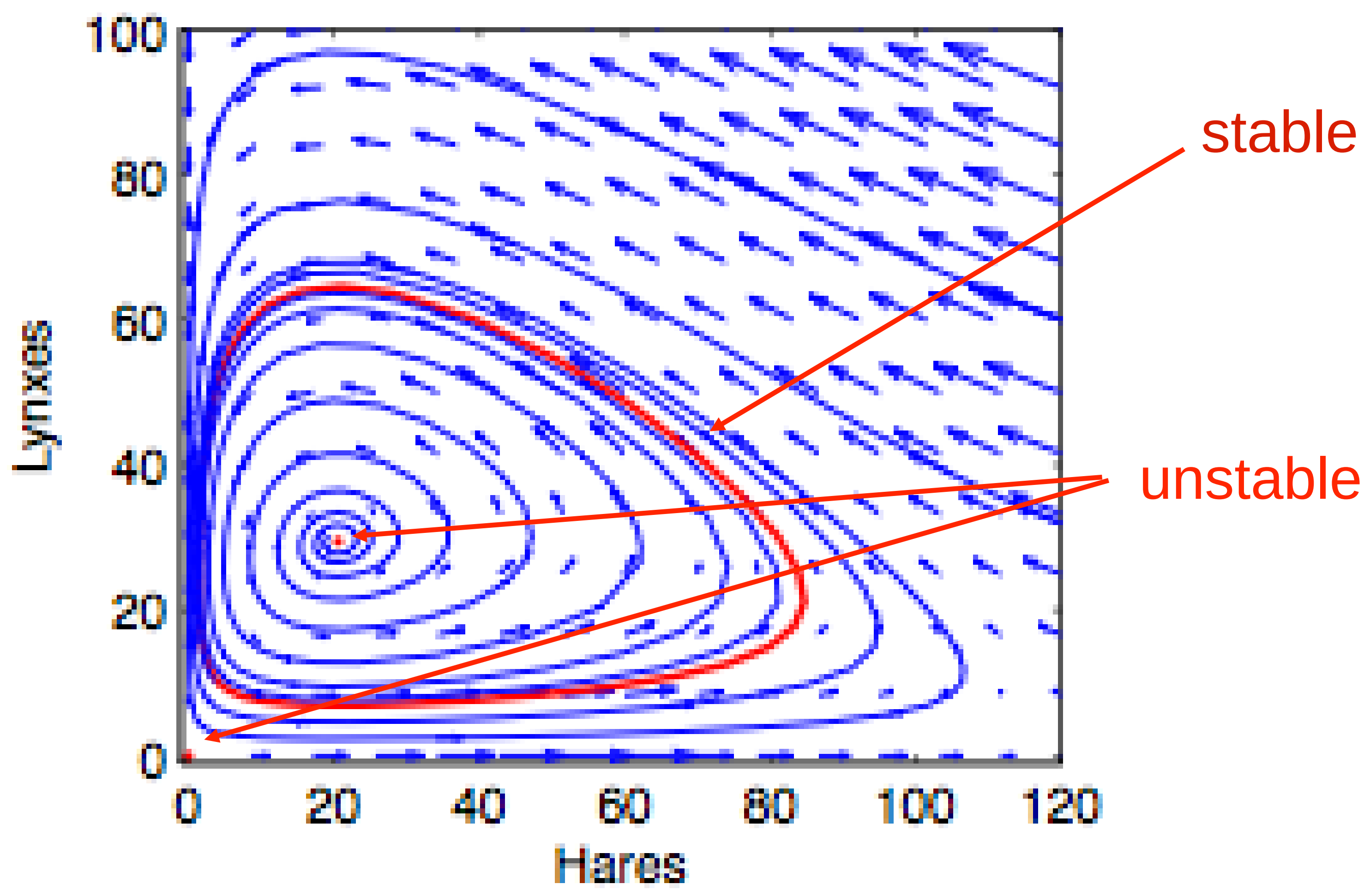
- Continuous time (ODE) model
- MATLAB: predprey.m (from web page)

Equilibrium points (2)

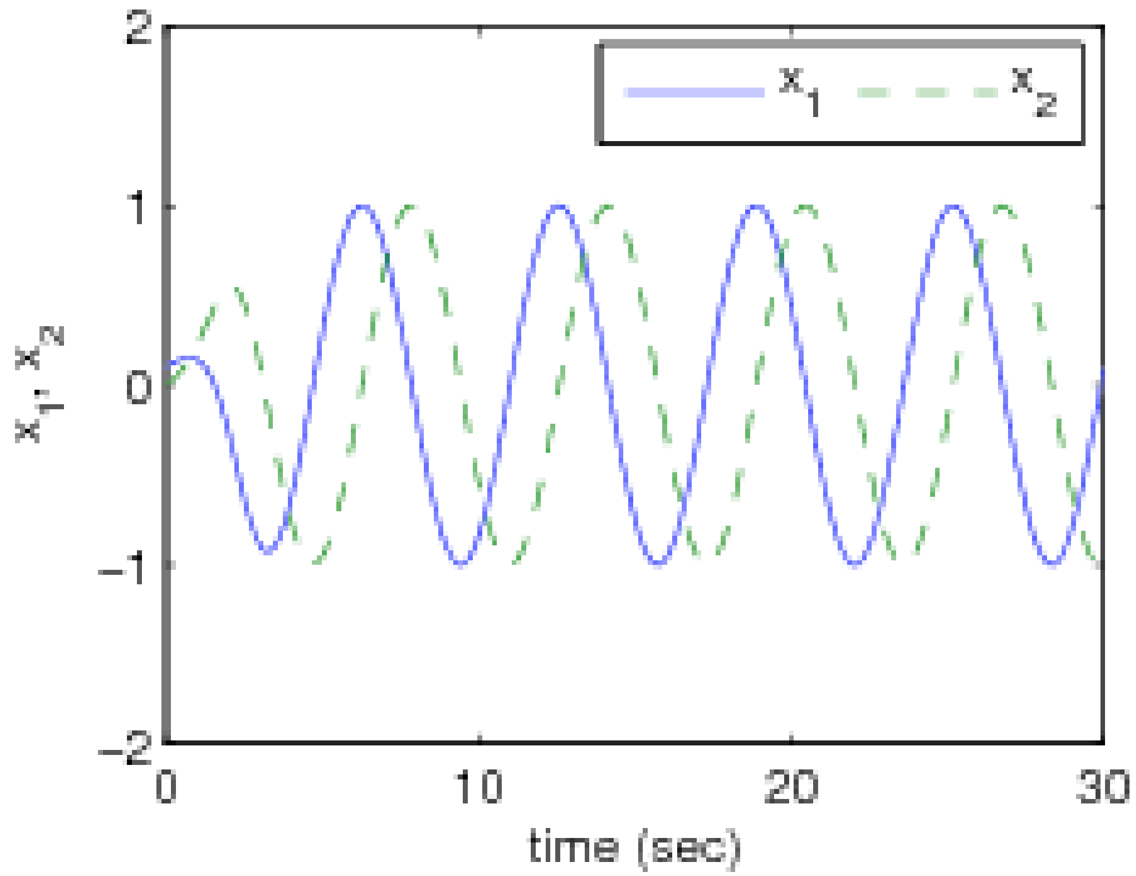
- ~(20.5, 29.5): unstable
- (0, 0): unstable

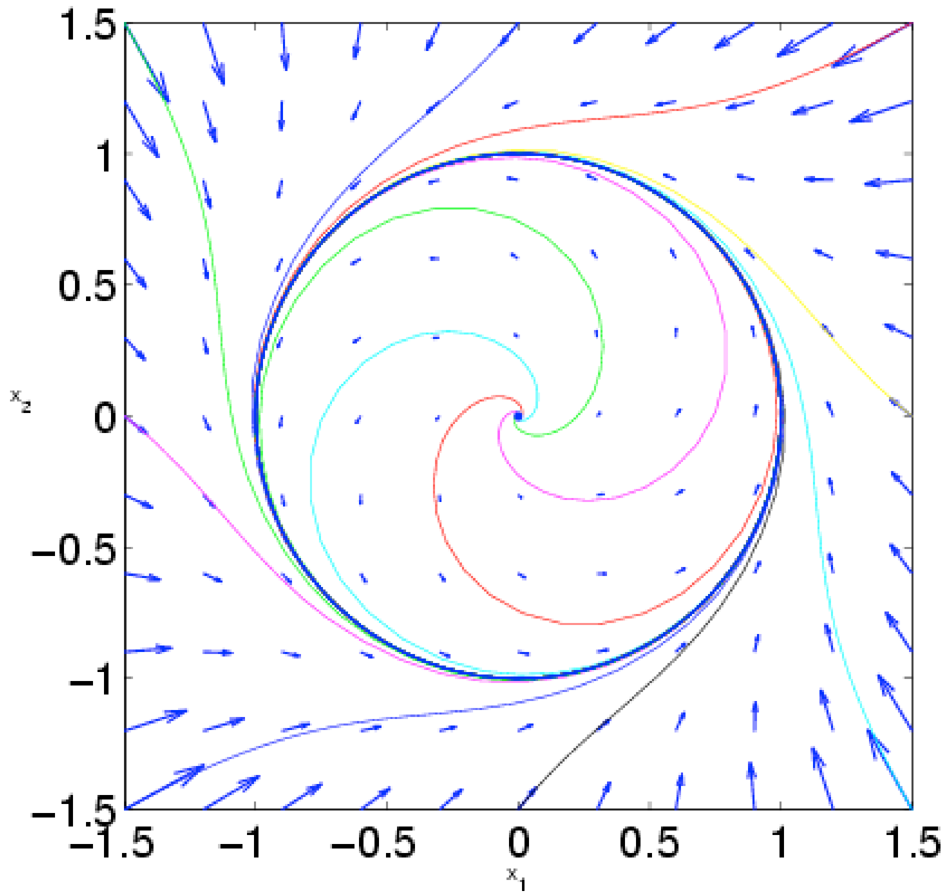
Limit cycle

- Population of each species oscillates over time
- Limit cycle is stable (nearby solutions converge to limit cycle)
- This is a *global* feature of the dynamics (not local to an equilibri point)



Simpler Example of a Limit Cycle





Dynamics:

$$\frac{dx_1}{dt} = -x_2 - x_1(1 - x_1^2 - x_2^2)$$
$$\frac{dx_2}{dt} = x_1 - x_2(1 - x_1^2 - x_2^2).$$

$$\|x\| = 1$$

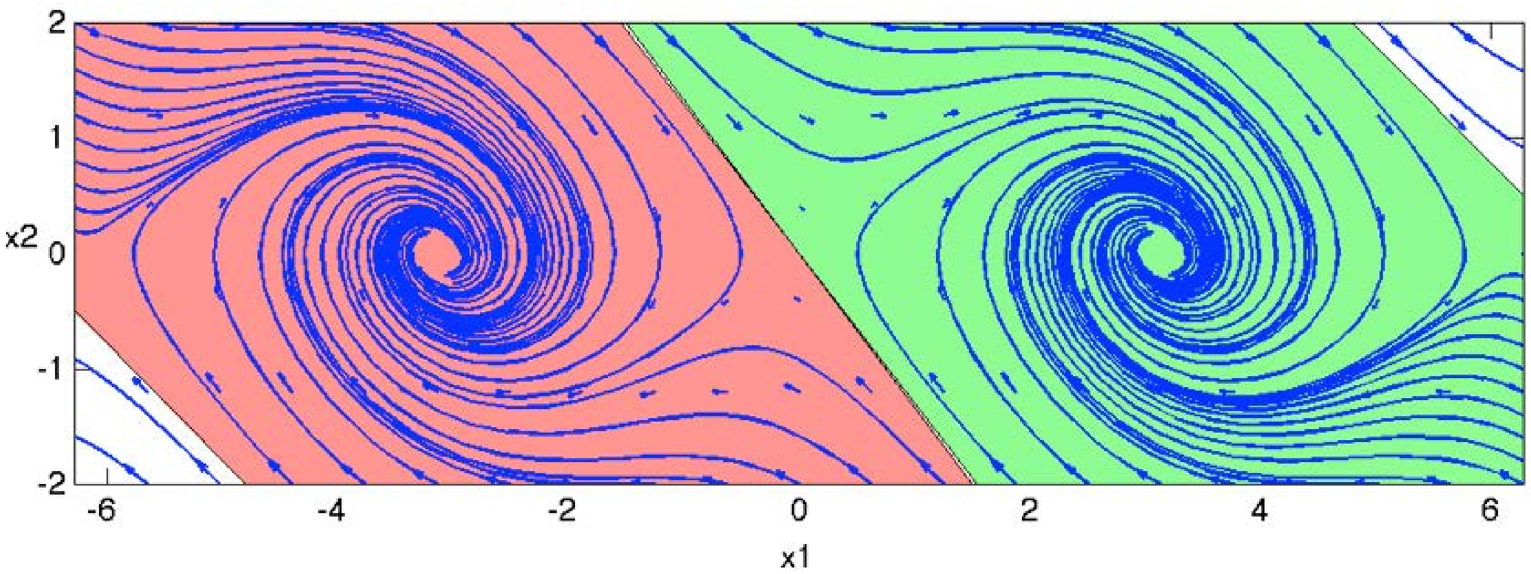
- Note that limit cycle is an *invariant set*
- From simulation, x(t+T) = x(t)

$$V(x) = \frac{1}{4}\left(1 - x_1^2 - x_2^2\right)^2$$

Stability of invariant set

$$\begin{aligned}\dot{V}(x) &= (x_1\dot{x}_1 + x_2\dot{x}_2)(1 - x_1^2 - x_2^2) \\ &= \cdots \\ &= -(x_1^2 + x_2^2)\left(1 - x_1^2 - x_2^2\right)^2\end{aligned}$$

Summary: Stability and Performance



Key topics for this lecture

- Stability of equilibrium points
- Eigenvalues determine stability for linear systems
- Local versus global behavior