CDS 112: Winter 2014/2015

The Brachistochrone Problem

Imagine a bead moving frictionlessly on wire or surface, described by $y(x)$, under the influence of gravity. The Brachistochrone problem is to find the path $y(x)$ that minimizes the bead's time of travel from a beginning point, A, to a final point, B. For sake of convenience locate point A at $(0,0)$ and point B at (B_x, B_y) (see Figure 1).

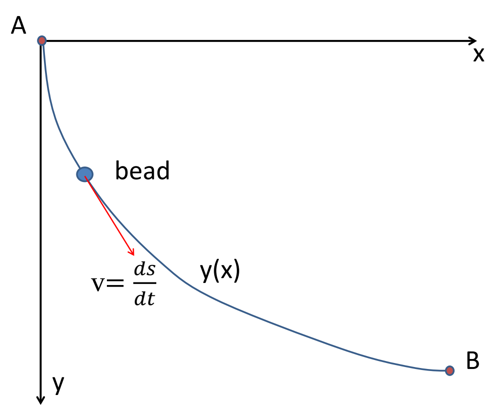

Figure 1: Geometry of the Brachistochrone problem

First, let's derive the cost function which will govern the Calculus of Variations problem which is solved to define the Brachistochrone curve. Let s denote the arc-length along the curve $y(x)$. The speed of the bead is given by $v = \frac{ds}{dt}$. Since the goal is to minimize the time taken reach B, the cost is:

$$Cost = \int_0^T \, dt \; = \; \int_0^{s_T} \, \frac{ds}{v} \; . \tag{1}$$

where s_T is the arc-length distance of the curve between points A and B. The speed can be expressed in terms of the projections of the bead velocity on the coordinate axes

$$\frac{ds}{dt} \; = \; \sqrt{(\frac{dx}{dt})^2 \; + \; (\frac{dy}{dt})^2} \; = \; \frac{dx}{dt}\sqrt{1 + \frac{dy}{dx}} \; . \tag{2}$$

Multiplying both sides of this relationship by dt yields:

$$ds \; = \; \sqrt{1 + (y')^2} \; dx \tag{3}$$

where $y^{'} = dy/dx$.

An expression for the velocity can be derived from the total energy of the bead's movement. Assume that the bead starts at rest at point A, where it's potential energy is chosen to be zero. Since the bead moves frictionlessly, the sum of it's potential and kinetic energy must be constant:

$$0 \;=\; m\frac{v^2}{2} - mgy(x) \qquad \Rightarrow \qquad v(x) = \sqrt{2gy(x)}\;. \tag{4}$$

Substituting the expressions for ds and v into Equation (1) yields:

$$Cost \;=\; \frac{1}{\sqrt{2g}}\int_0^{B_x} \sqrt{\frac{1+(y')^2}{y}}\; dx \;\triangleq\; \int_0^{B_x} \;\mathcal{L}(y,y^{'})\; dx \tag{5}$$

Normally, one would substitute the Lagrangian, $\mathcal{L}(y,y^{'})$ into the Euler-Lagrange equation

$$\frac{d}{dx}\left[\frac{\partial\mathcal{L}}{\partial y'}\right] \;-\; \frac{\partial\mathcal{L}}{\partial y} = 0 \tag{6}$$

to derive the differential equation would would describe the Brachistochrone. Because the cost function is *not* an explicit function of x, we can use the *Beltrami Identity* to simplify the calculations. The Beltrami Identity is derived as follows.

$$\begin{aligned} \frac{d}{dx}\left(\mathcal{L}(y,y^{'}) - y^{'}\frac{\partial\mathcal{L}}{\partial y'}\right) \;&=\; \frac{\partial\mathcal{L}}{\partial y}y^{'} + \frac{\partial\mathcal{L}}{\partial y'}y^{''} - y^{''}\frac{\partial\mathcal{L}}{\partial y'} - y^{'}\frac{d}{dx}\left[\frac{\partial\mathcal{L}}{\partial y'}\right] &(7)\\ &=\; y^{'}\left(\frac{\partial\mathcal{L}}{\partial y} - \frac{d}{dx}\left[\frac{\partial\mathcal{L}}{\partial y'}\right]\right) &(8)\end{aligned}$$

However, since the term in the parentheses is the Euler-Lagrange equation, which must be zero, this calculation implies that:

$$\mathcal{L} \;-\; y^{'}\frac{\partial\mathcal{L}}{\partial y'} = C\;. \tag{9}$$

Substituting in the expression for $\mathcal{L}$ and simplifying yields:

$$\sqrt{2g}\sqrt{y(1+(y')^2)} \;=\; \frac{1}{C} \tag{10}$$

Squaring both sides and rearranging results in:

$$y(1+(y^{'})^2) = \frac{1}{2gC^2} = k^2 \tag{11}$$

where undetermined constant k subsumes the constant of integration and gravity.

Next, let's reparametrize the curve $y(x)$ to simplify the integration. Let

$$y^{'} = \tan\phi \tag{12}$$

where ϕ is the new parametrization of the Brachistochrone curve. Substituting (12) into (11) and rearranging yields:

$$y \;=\; k^2\cos^2\phi\;. \tag{13}$$

To proceed, next differentiate the expression for y, and use the new identity that $y^{'} = \tan\phi$

$$\frac{dy}{dx} = -2k^2\frac{d\phi}{dx}\sin\phi\ \cos\phi\ = \tan\phi\ . \tag{14}$$

Divide both sides by $\tan\phi$ and rearrange to yield:

$$dx = -k^2(1+\cos(2\phi))d\phi \tag{15}$$

Integrate (15) to get

$$x = C_1 - \frac{k^2}{2}(2\phi + \sin(2\phi)) \tag{16}$$

Returning to (13), we see that $y = \frac{k^2}{2}\cos^2\phi = \frac{k^2}{2}(1+\cos(2\phi))$, which then completes the parameterization of the curve in terms of a parameter $t = 2\phi$:

$$x(t) = C_1 - \frac{k^2}{2}(t + \sin t) \tag{17}$$

$$y(t) = \frac{k^2}{2}(1+\cos t) \tag{18}$$

These equations describe a *cycloid*. This can be seen by introducing another reparametrization of the curve: $t = -z + \pi$, which puts the equations in a more standadard form:

$$x(t) = C_2 + \frac{k^2}{2}(z - \sin z) \tag{19}$$

$$y(t) = \frac{k^2}{2}(1-\cos z) \tag{20}$$

where $C_2 = (C_1 - k^2\pi/2)$ The parameters C_1, k, and the boundaries of the interval $[z_0, z_T]$ can then be found by using the boundary conditions at points A and B.