# CDS 112 Final Exam

(Winter 2014/2015)

## Instructions

1. Limit your total time to 5 hours. It is okay to take a break in the middle of the exam if you need to ask the Instructor or TA a question, or to go to dinner. If you run out of time, indicate how you would proceed as explicitly as possible.

2. You may use any class notes, books, or other written material posted on the course web site. You may not discuss this final with other class students or other people except me or the class Teaching Assistants.

3. You may use Mathematica, MATLAB, or any software or computational tools to assist you.

4. You cannot use the internet to solve these problems, except for material on the course web site.

5. The final is due by 5:00 p.m. on the last day of finals.

6. The point values are listed for each problem to assist you in allocation of your time.

**Problem 1:** Control of a two-wheeled mobile robot

Consider the two wheeled mobile robot shown in Figure 1

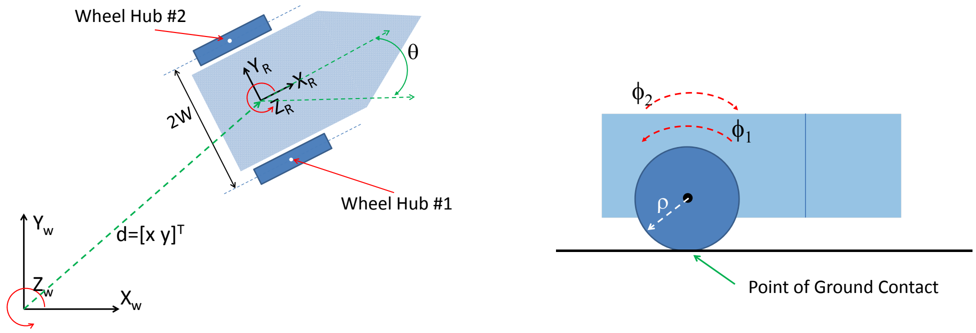


Figure 1: **Left:** Top view of planar two-wheeled robot; **Right:** Side view of planar two-wheeled robot

**Part (a):** (15 points) Consider the problem of moving the cart from one initial position and orientation, $(x_0, y_0, \theta_0)$, to a final configuration $(x_f, y_f, \theta_f)$ in minimum time. As show in the figure, The $(x, y)$ coordinates describe the location of the mid-point of the cart's axel in the plane, while $\theta$ denoted the cart's orientation relative to the fixed frame. The governing equations of motion for this vehicle can be simplified to the following form

$$\dot{x} \;=\; v\cos\theta \qquad \dot{y} \;=\; v\sin\theta \qquad \dot{\theta} = \omega$$

where the control inputs $(v, \omega)$ are respectively the forward velocity of the vehicle (it's velocity along its body-fixed $x_R$ axis) and its spinning rate. The control are subject to the following constraints:

$$|v| \le L \qquad |\omega| \le M \ .$$

Use the maximum principle to show that any optimal trajectory consists of segments in which the robot is traveling at maximum velocity in the forward or reverse direction, and is moving along a local trajectory segment that is straight, hard left ($\omega = M$), or hard right ($\omega = -M$).

**Part (b):** (25 points) This part of the problem will consider how to stabilize the non-linear system you analyzed in part (a) about a *trajectory* using the LQR methodology. We want to design a state feedback law that allows the vehicle to track an arbitrary trajectory, specified by $\bar{x}(t)$, $\bar{U}(t)$

1. (5 points) Linearize this system about the trajectory $\bar{x}(t)$, $\bar{U}(t)$, and perform a coordinate change to get a new (*linear*) system of the form

$$\dot{z} = A(t)z + B(t)\phi$$

*Hint:* Recall the rules/recipe for linearization about an equilibrium point from CDS 110. The only difference in this case is that your 'equilibrium' is a function of time.

2. (5 points) Obtain a state feedback law of the form $\phi = -K(t)z$ that minimizes

$$J = \int_0^\infty z^T Q z + \phi^T R \phi \; dt$$

in terms of $A(t)$, $B(t)$, $Q$, $R$. Write the corresponding control law for the original, non-linear system (i.e. write down $U(t)$ in terms of $\bar{U}(t)$, $K(t)$ $z(t)$).

3. (15 points) For initial conditions $x(0) = [-5, -7, -\frac{\pi}{2}]^T$, and

$$Q = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \beta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \bar{x}(t) = \begin{pmatrix} t\cos\frac{\pi}{4} \\ t\sin\frac{\pi}{4} \\ \frac{\pi}{4} \end{pmatrix}, \quad \bar{U}(t) = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

apply you LQR control law to the original *non-linear system* and plot the resulting paths for $0 \le t \le 10$ with $\alpha = 1$, $\beta = 1$.

(*Hint:* Use `lqr` and `ode45` in `MATLAB`. )

4. (**5 points extra credit**) Repeat your plot for the case $\alpha = 1$, $\beta = 50$, and describe how varying $\alpha$ and $\beta$ affect tracking. Explain the effects in terms of the cost, $J$.

**Problem :** (Different noise models in Kalman Filtering)

**Part (a):** (15 Points) Consider the standard linear dynamical system set-up (we will ignore the control inputs for simplicity):

$$\begin{aligned} x_{k+1} &= F x_k + G \eta_k \\ y_{k+1} &= H x_{k+1} + \omega_{k+1} \end{aligned} \tag{1}$$

with Gaussian, zero mean white process and measurement noises:

$$E \begin{bmatrix} \eta_k \\ \omega_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

However, there is now correlation between the process and measurement noises:

$$E\left\{ \begin{bmatrix} \eta_{k-1} \\ \omega_k \end{bmatrix} \begin{bmatrix} \eta_{k-1}^T & \omega_k^T \end{bmatrix} \right\} = \begin{bmatrix} Q_{k-1} & S_k \\ S_k^T & R_k \end{bmatrix} .$$

1. (5 Points) Show that the measurement update covariance for systems with this type of correlated process and measurement noise takes the form:

$$\Sigma_{k|k} = (I - K_k H_k)\Sigma_{k|k-1}(I - K_k H_k)^T + K_k R_k K_k^T + K_k(H_k S_k + S_k^T H_K^T)K_k^T - S_k K_k^T - K_k S_k^T \tag{2}$$

(*Hint: write the measurement state estimate update equation as* $\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(y_k - H_k \hat{x}_{k|k-1})$, *and use this equation to derive the covariance update* ).

2. (10 Points) Find the expression for the Kalman gain matrix, $K_k$, which minimizes the variance of the estimate.

**Part (b):** As we discussed in class, the white noise assumption is a convenient fiction which is useful for purposes of deriving estimators with simple structure. Let's consider incorporating a more realistic noise model into the measurement equation:

$$\begin{aligned} x_{k+1} &= Fx_k + G\eta_k \\ y_{k+1} &= Hx_{k+1} + n_{k+1} \end{aligned} \tag{3}$$

where $n_j$ is the perturbing measurement noise at time $t_j$. We assume that the measurement noise is the result of the following *Gauss-Markov Process*:

$$n_{k+1} = \Psi_k n_k + \omega_k$$

where $\omega_{k-1}$ is zero mean, white, Gaussian noise process. The structure and values of the elements comprising $\Psi$ can define different types of "colored" noise. We now want to derive the appropriate Kalman Filtering equations for this new type of system.

1. (5 Points) A simple way to handle this problem is to define an *augmented* state vector, consisting of $\vec{z}_k = [x_k \ n_k]^T$. The dynamics of this augmented system are simply:

$$\begin{bmatrix} x_{k+1} \\ n_{k+1} \end{bmatrix} = \begin{bmatrix} F_k & 0 \\ 0 & \Psi_k \end{bmatrix} \begin{bmatrix} x_k \\ n_k \end{bmatrix} + \begin{bmatrix} G \\ I \end{bmatrix} \begin{bmatrix} \eta_k \\ \omega_k \end{bmatrix}$$

   What is a potential problem with this approach?

2. (10 Points) A better solution was developed by Bryson and Hendrikson[1]. They suggest the following *measurement differencing* approach to produce an *equivalent measurment*, $\xi$:

$$\xi_{k-1} = y_k - \Psi_{k-1} y_{k-1}$$

   where $y_k$ is the measurement at $t_k$.

   - Show that this measurement equation is equivalent to

$$\xi_{k-1} = D_{k-1} x_{k-1} + \gamma_{k-1}$$

     where

$$\begin{aligned} D_{k-1} &= H_k F_{k-1} - \Psi_{k-1} H_{k-1} \\ \gamma_{k-1} &= H_k \eta_{k-1} + \omega_{k-1} \end{aligned}$$

   - Show that

$$E\left\{ \begin{bmatrix} \eta_{k-1} \\ \gamma_{k-1} \end{bmatrix} \begin{bmatrix} \eta_{k-1}^T & \gamma_{k-1}^T \end{bmatrix} \right\} = \begin{bmatrix} Q_{k-1} & Q_{k-1} H_k^T \\ H_k Q_{k-1} & (H_k Q_{k-1} H_k^T + R_{k-1}) \end{bmatrix}$$

     and thus one can use the Kalman Filtering update equations derived in Part (a) of this problem to filter a system whose colored noise results from a Gauss-Markov process.

[1] A.E. Bryson Jr. and I.J. Henrikson, "Estimation Using Sampled Data Containing Sequentially Correlated Noise," *AIAA J. Spacecraft and Rockets*, vol. 5, no. 6, pp. 662-665, June 1968.