CDS 112: Winter 2014/2015

Homework #1: due Friday, January 16, 2015

Problem #1: In class we derived the optimal bang-bang control for a system consisting of a bead of mass m pushed along a wire by thrust u, with u bounded: $|u| \leq C$. This problem will consider some variations of this problem. Again, consider a bead of mass $m = 1$ constrained to move frictionlessly along a wire that is collinear with the x-axis of a Cartesion coordinate system. Hence, the dynamics take the very simple form:

$$\ddot{x} \;=\; u\;.$$

For all of the variations of this problem, assume that the goal is to bring the bead to rest at the origin: $x(T) \;=\; \dot{x}(T) = 0$, where T is the final time.

- **Part (a):** Find the control that brings the bead from its initial state ($t_0 = 0$; $x(t_0) = x_0$; $\dot{x}(t_0) = 0$) to rest (zero final velocity) at the origin of the x-axis in minimum time. The control u is bounded in this case as follows.: $-2 \leq u \leq 1$. I.e., the upper and lower control limits are not symmetrical.
- **Part (b):** Let's return to the case of a symmetric bound on the control: $|u| \leq 1$. For the initial conditions, $x(t_0) = x_0$ and $\dot{x}(t_0) = 0$, find the control which minimizes the following objective function:

  $$J(x, u) \;=\; \int_0^T 1 \;+\; \alpha u^2 \; dt \qquad (1)$$

  where α is a positive constant.
- **Part (c):** Now consider the goal of minimize the *energy* used by the system to go from the initial conditions $x(t_0) = 1$ and $\dot{x}(t_0) = 1$ to a state of rest at the origin. The goal is to minimize the cost function:

  $$J(x, u) \;=\; \int_0^1 u^2(t)\; dt.$$

  You need not assume any constraints on the controls.

Problem #2: In class, and in the notes, we derived, using a "variational" approach, the ordinary differential equations defining the optimal control in the case of an optimal control problem have governing equations:

$$\dot{x} = f(x, u)$$

and cost function:

$$J(x, u) \;=\; \int_0^T L(x, u)\; dt \;+\; V(x(T))$$

where x is the system state, u is the system control input, $L(x, u)$ is the instantaneous cost of the control, and $V(x(T))$ is a terminal penalty function on the terminal system state $x(T)$ at terminal time T. The ordinary differential equations were derived under the assumption that the intial state, x_0 was specified, but the final state x_F was not specified. We also assumed that the final time T was given. In this problem, you are to derive the additional constraints that occur when:

- **Part (a):** the terminal state is given. I.e., the state at time T is X_F. In particular, a terminal state function $\psi(x_F) = 0$ is specified.
- **Part (b):** the terminal time is not specified.