CDS 112: Winter 2014/2015

Solution to Problem Set #1

Solutions to Problem 1(a),(b),(c) M.Burkhardt Jan. 2015

Problem 1(a) Statement: Find the control that brings the bead from its initial state ($t_0 = 0$; $x(t_0) = x_0$; $\dot{x}(t_0) = 0$) to rest (zero final velocity) at the origin of the x axis in minimum time. The control u is bounded in this case as follows: $-2 \leq u \leq 1$ i.e. the upper and lower control limits are not symmetrical.

Our state dynamics for parts (a) and (b) are:

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1}$$

Since our problem is a minimum time problem, our cost functional is:

$$\int_{t_0}^{T} 1 \, dt \tag{2}$$

We can then form the Hamiltonian as: $H = 1 + \lambda^T(A\vec{x} + Bu)$, where A and B are written from (1). Written in different terms, $H = 1 + \lambda_1 \dot{x} + \lambda_2 u$. At this point we can turn to the PMP conditions. First, invoke the condition often called 'Pontryagin's Minimum Principle', i.e. $H(u^*, x^*, \lambda^*) \leq H(u, x^*, \lambda^*)$ for all (t, u^*, λ^*, x^*). This condition will give the form of the controller:

$$u^* = \begin{cases} -2 & : sign(\lambda_2) > 0 \\ 1 & : sign(\lambda_2) < 0 \end{cases} \tag{3}$$

The PMP condition governing the co-state's differential equation, i.e. $-\dot{\lambda} = \frac{\partial H}{\partial x}$ yields:

$$-\dot{\lambda_1}(t) = 0 \Rightarrow \lambda_1 = const = C_1 \tag{4}$$

$$-\dot{\lambda_2} = C_1 \Rightarrow \lambda_2 = C_1 t + C_2 \tag{5}$$

This problem has a terminal constraint set, $\psi(\vec{x}(T)) = \vec{x}(T) = 0$, so invoking the PMP condition that specifies the final co-state value yields $\lambda_1(T) = \nu_1$ and $\lambda_2(T) = \nu_2$, and combining this result with equations (4) and (5) gives:

$$\lambda_2(t) = \nu_1(t - T) + \nu_2 \tag{6}$$

We can obtain more structure by using the PMP condition pertaining to an undetermined final time: $H(T) = 0$. This yields $1 + \nu_2 u^*(T) = 0$. If $\nu_2 > 0$, then $u^*(T) = -2 \Rightarrow \nu_2 = 1/2$, and if $\nu_2 < 0 \Rightarrow \nu_2 = -1$. At this point, we can use the fact that for this problem we must end up at the origin, which implies that exactly one switch occurs in the control. Also, due to the asymmetry in the control constraints, we would expect an asymmetrical answer, so for now let's assume $x_0 > 0$ and we will handle the other case later. Let t_s denote the switching time and integrate the dynamics $\ddot{x} = u$ to obtain:

$$\dot{x}(T) = \int_{t_0=0}^{t_s} -2 \, dt + \int_{t_s}^{T} 1 \, dt + \dot{x}(0) \Rightarrow t_s = T/3 \tag{7}$$

Now integrate the dynamics between $[0, T/3]$ and $[T/3, T]$, and use the condition that $x(T) = 0$ to obtain the result that for $x_0 > 0 \Rightarrow T = \sqrt{3x_0}$. Now we can fully determine the controller. Using the fact that $\lambda_2(t_s) = 0$, we can relate ν_1 to ν_2 via:

$$(2T/3)\nu_1 = \nu_2 \tag{8}$$

And, for the case that $x_0 > 0$ we know that $u^*(T) > 0 \Rightarrow sign(\lambda_2(T)) < 0 \Rightarrow \nu_2 = -1$, and correspondingly $\nu_1 = -(3/2T)$. Putting it all together, the deterministic controller for the case $x_0 > 0$ is given by:

$$\boxed{u^* = \begin{cases} -2 & : -(3/2T) * (t - \sqrt{3 * x_0}) - 1 > 0 \\ 1 & : -(3/2T) * (t - \sqrt{3 * x_0}) - 1 < 0 \end{cases}} \tag{9}$$

So, what changes when $x_0 < 0$? Equations (1-6) remain valid, and the expectation of one switching time is also valid. Equation (7) is rewritten to be:

$$\dot{x}(T) = \int_{t_0=0}^{t_s} 1 \; dt + \int_{t_s}^{T} -2 \; dt + \dot{x}(0) \Rightarrow t_s = 2T/3 \tag{10}$$

Again, integrating the dynamics between $[t_0, 2T/3]$ and $[2T/3, T]$, and using the fact that $x(T) = 0$ will yield the final time as $T = \sqrt{-3x_0}$, which is not alarming due to the fact that $x_0 < 0$. Now, using the condition $\lambda_2(t_s) = 0$ we can relate ν_1 to ν_2 via:

$$\nu_1 T/3 = \nu_2 \tag{11}$$

Knowing that $u^*(T) < 0 \Rightarrow sign(\lambda_2(T)) > 0 \Rightarrow \nu_2 = 1/2$, which then implies that $\nu_1 = 3/(2T)$. Assembling the controller, which has been fully specified we find that:

$$\boxed{u^* = \begin{cases} -2 & : (3/2T) * (t - \sqrt{-3 * x_0}) + 1/2 > 0 \\ 1 & : (3/2T) * (t - \sqrt{-3 * x_0}) + 1/2 < 0 \end{cases}} \tag{12}$$

Problem 1(b) Statement: Let's return to the case of a symmetric bound on the control: $|u| \leq 1$. For the initial conditions, $x(t_0) = x_0$ and $\dot{x}(t_0) = 0$, find the control which minimizes the following objective function: $\int_0^T 1 + \alpha u^2 \; dt$, where $\alpha > 0$. As mentioned in the office hours, we are only looking for the structural form of the controller, not for the controller to be fully specified. We can arrive at this by forming the Hamiltonian and applying Pontryagin's Minimum Principle.

$$H = 1 + \alpha u^2 + \lambda_1 \dot{x} + \lambda_2 u \tag{13}$$

Note that the only extra terms on the Hamiltonian result from the revised instantaneous cost $L(x, u)$ between part (a) and part (b). Turning to the PMP condition given above, we can specify the form of the controller by choosing the optimal controller u^* such that:

$$1 + \alpha(u^*)^2 + \lambda_1^* \dot{x}^* + \lambda_2^* u^* \leq 1 + \alpha u^2 + \lambda_1^* \dot{x}^* + \lambda_2^* u \tag{14}$$

Due to the fact that the LHS of (14) is quadratic in u^*, we can simply minimize the LHS with respect to u^* while respecting the symmetrical constraints on the control. This is justified since $\alpha > 0$. So, setting the derivative of the LHS with respect to u^* equal to zero we obtain:

$$u^* = \boxed{\begin{cases} -\lambda_2/(2*\alpha) & : |\lambda_2| < 2\alpha \\ -1 & : \lambda > 2\alpha \\ 1 & : \lambda < -2\alpha \end{cases}} \tag{15}$$

Why is the controller difficult to specify in full generality? This controller is difficult to specify in general because depending on which part of the phase space we start in either one or two control switches may be required (one switch from positive to negative within the linear portion of the controller, and one if a saturation limit is reached). Thus, to fully pin down this control, we have to enumerate all possible cases, which is tedious and not in the spirit of the assignment.

Problem 1(c) Statement: Now consider the goal of minimizing the energy used by the system to go from the initial conditions $x(t_0) = 1$ and $\dot{x}(t_0) = 1$ to a state of rest at the origin. The goal is to minimize the cost function: (You need not assume any constraints on the cost function).

$$J(x,u) = \int_0^{T=1} u^2(t)\ dt \tag{16}$$

As always, we formulate the Hamiltonian, $H = u^2 + \lambda_1 \dot{x} + \lambda_2 u$. Since there are no restrictions on the control, we can let $\frac{\partial H}{\partial u} = 2u^*(t) + \lambda_2(t) = 0 \Rightarrow u^*(t) = -\frac{1}{2}\lambda_2(t)$. Turn to the PMP condition $\frac{\partial H}{\partial x} = -\dot{\lambda}(t) \Rightarrow \lambda_2(t) = \nu_1(t-T) + \nu_2$, where $\lambda_1(t) = \nu_1$.

Now that we have the controller specified as a function of ν_1 and ν_2, we can use the system dynamics to derive equations to determine these constants.

$$\dot{x}(t) = \frac{1}{2}(\nu_1 T - \nu_2)t - \frac{1}{4}\nu_1 t^2 + 1 \tag{17}$$

$$x(t) = \frac{1}{4}(\nu_1 T - \nu_2)t^2 - \frac{1}{12}\nu_1 t^3 + t + 1 \tag{18}$$

We can then form a system of equations for solve for these two constants using the terminal conditions provided, under the specialization that $T = 1$. Solving for this system of equations yields $\nu_1 = -36$ and $\nu_2 = -16$. Pluggin this back into our equation for the controller yields:

$$\boxed{u^* = 18t - 10} \tag{19}$$

Problem #2: In class, and in the notes, we derived, using a "variational" approach, the ordinary differential equations defining the optimal control in the case of an optimal control problem have governing equations:

$$\dot{x} = f(x,u)$$

and cost function:

$$J(x,u) \ = \ \int_0^T L(x,u)\ dt \ + \ V(x(T))$$

where x is the system state, u is the system control input, $L(x,u)$ is the instantaneous cost of the control, and $V(x(T)$ is a terminal penalty function on the terminal system state $x(T)$ at terminal time T. The ordinary differential equations were derived under the assumption that the intial state, x_0 was specified, but the final state x_F was not specified. We also assumed that the final time T was given. In this problem, you are to derive the additional constraints that occur when:

- **Part (a):** the terminal state is given. I.e., the state at time T is X_F. In particular, a terminal state function $\psi(x_T) = 0$ is specified.
- **Part (b):** the terminal time is not specified.

Solution (a): Our goal is to construct the variational of the appropriate cost function, and show that the vanishing of the variation leads to the desired conditions. The standard integral cost must be augmented with two underdertermined Lagrange multiplier terms, one for the dynamics constraint, $\dot{x} = f(x, u)$, and one for each of the k scalar terminal constraints, $\psi_i(x_T)$, $i = 1, \ldots, k$:

$$\tilde{J} = V(x(T)) + \sum_{i=1}^{k} \nu_i \psi_i(x(T)) + \int_0^T [L(x,u) + \lambda^T (f(x,u) - \dot{x})] dt$$

Following the procedure in the *Optimization Based Control* notes, which was also reviewed in class, construct the variation of $\tilde{J}$, denoted $\delta\tilde{J}$:

$$\begin{aligned}
\delta\tilde{J} &= \tilde{J} - \tilde{J}^* = \tilde{J}(x^* + \delta x, u^* + \delta u, \lambda^* + \delta\lambda, x^*(T) + \delta x(T)) - \tilde{J}(x^*, u^*, \lambda^*) \\
&= V(x^*(T) + \delta x(T)) - V(x(T)) + \sum_{i=1}^{k} \nu_i \psi_i(x(T) + \delta x(T)) - \sum_{i=1}^{k} \psi(x(T)) \\
&+ \int_0^T \left[H(x^* + \delta x, u^* + \delta u, \lambda^* + \delta\lambda) - (\lambda^* + \delta\lambda)^T \frac{d}{dt}(x^* + \delta x) \right] dt \\
&- \int_0^T [H(x^*, u^*, \lambda^*) - (\lambda^*)^T \dot{x}^*] dt \\
&= \left(\frac{\partial V}{\partial x}\right)^T \delta x(T) + \sum_{i=1}^{k} \nu_i \left(\frac{\partial \psi}{\partial x}(x(T))\right)^T \delta x(T) \\
&+ \int_0^T \left[\left(\frac{\partial H}{\partial x}\right)^T \delta x + \left(\frac{\partial H}{\partial u}\right)^T \delta u + \left(\frac{\partial H}{\partial \lambda}\right)^T \delta\lambda - (\lambda^*)^T (\dot{\delta x}) - (x^*)^T \delta\lambda - (\dot{\delta x})^T \delta\lambda \right] dt
\end{aligned}$$

where $(\dot{\delta x})$ is short-hand notation for $\frac{d}{dt}(\delta x)$. This equation can be simplified by first noting that the term involving the product of variations will vanish in the limit of small variations, and the term involving $\frac{d}{dt}(\delta x)$ can be rewritting using integration by parts:

$$\int_0^T (\lambda^*) \frac{d}{dt}(\delta x) dt = (\lambda^*)^T(0)\delta x(0) + (\lambda^*(T))^T \delta x(T) - \int_0^T (\lambda^*)^T \delta x dt$$

Substituing in these simplifications, and combining terms yields:

$$\begin{aligned}
\delta\tilde{J} &= \left[\sum_{i=1}^{k} \nu_i \left(\frac{\partial \psi}{\partial x}\right)^T - \lambda^T + \left(\frac{\partial V}{\partial x}\right)^T \right] \delta x(T) - \lambda^T(0)\delta x(0) \\
&+ \int_0^T \left[\left[\left(\frac{\partial H}{\partial x}\right)^T + (\dot{\lambda})^T \right] \delta x + \left(\frac{\partial H}{\partial u}\right)^T \delta u + \left[\left(\frac{\partial H}{\partial \lambda}\right)^T - (\dot{x})^T \right] \delta\lambda \right] dt
\end{aligned}$$

Since we assume that the initial condition is fixed, $\delta x(0) = 0$. For the variation to be stationary with respect to all small variations, then the terms in brackets before the variations must vanish. This leads to the standard optimal control equations, plus the additional constraint:

$$\lambda^T = \sum_{i=1}^{k} \nu_i \left(\frac{\partial \psi_i}{\partial x}\right)^T + \left(\frac{\partial V}{\partial x}\right)^T$$

Transposing the equation and denoting $\vec{\nu} = [\nu_1, \ldots, \nu_k]^T$, yields:

$$\vec{\lambda} = \frac{\partial V}{\partial x}(x(T)) + \left(\frac{\partial \vec{\psi}}{\partial x}\right)_{x=x(T)} \vec{\nu}.$$

Solution (b): When the final time, T, is not fixed, it is necessary that the cost be stationary with respect to variations in the final time. Building upon the derivation in **Part (a)**, it should be clear that in the case of variable final time, the ***total variation*** of $\tilde{J}$ should vanish, where now we must consider variations in both $x(t)$, $u(t)$ $\lambda(t)$, as well as the variation in $\tilde{J}$ with respect to time:

$$D\tilde{J} = \delta\tilde{J}(T^*) + \frac{dJ}{dt}\Big|_{t=T^*} = 0.$$

Since we know that $\delta\tilde{J}(T^*) = 0$ for variations in $x(t)$, $u(t)$ $\lambda(t)$, we additionally require that $\frac{dJ}{dt} = 0$ at the final time. The actual form of the cost function $\tilde{J}$ depends upon the problem conditions. Let's consider the case of a terminal penalty cost, but no terminal constraint:

$$\tilde{J} = V_T(x(T)) + \int_0^T \left[H(u(t), x(t), y(t) - \lambda^T \dot{x}\right] dt\ .$$

Hence,

$$\frac{d\tilde{J}}{dt}\Big|_{t=T^*} = \left[\frac{dV_T}{dt} + \left(\frac{\partial V}{\partial x}\right)^T \dot{x} + H(u(t), x(t), y(t)) - \lambda^T \dot{x}\right]\Big|_{t=T^*} = 0. \qquad (20)$$

But since the vanishing of $\delta\tilde{J}$ implies that

$$\lambda = \frac{\partial V}{\partial x}$$

Equation (20) simplifies to:

$$\left[\frac{dV_T}{dt} + H(x, u, \lambda)\right]_{t=T^*} = 0.$$

If the terminal penality V_T is not an explicit function of time, then the condition simplifies to $H(x, u, \lambda)(T^*) = 0$