CDS 112: Winter 2014/2015

Homework #2: due Monday, Feb. 2, 2015

Problem #1: (25 points) This problem explores the difference between a simple finite horizon optimal control problem and its infinite horizon optimal counterpart. You may want to familiarize yourself with the `lqr` and `ode45` functions in MATLAB.

Consider a system whose dynamics are governed by the equations:

$$\dot{x} \;=\; ax \;+\; bu \tag{1}$$

where $x \in \mathbb{R}$ denotes the state, $u \in \mathbb{R}$ is a single scalar control input, and a, b are constant, *positive* scalars:

$$a = 2 \qquad b = 0.5\ . \tag{2}$$

Consider the optimal control problem with cost

$$J \;=\; \frac{1}{2}\int_{t_0}^{T} u^2(t)\ dt \;+\; \frac{1}{2}c\ x^2(T) \tag{3}$$

where final time T is given, and $c > 0$ is a constant. The optimal control for finite time $T > 0$ is derived in Example 2.2 of the optimal control notes. Now consider the infinite horizon problem with infinite horizon cost:

$$J \;=\; \frac{1}{2}\int_{t_0}^{\infty} u^2(t) + cx(t)^2\ dt \tag{4}$$

part (a): (5 points) Solve the algebraic Riccati equation to find P, leading to the optimal control $u^*(t) \;=\; -bPx^*(t)$ for the infinite horizon case.

part (b): (7 points) For the two initial conditions $x(t_0) = 0.1$ and $x(t_0) = 10.0$, plot the closed loop system response of this system designed for an infinite horizon over an interval of 10 seconds.

part (c): (10 points) Plot the closed loop system response of the *finite horizon* optimal controller for the case of $c = 0.1$ and $c = 10.0$. Plot the response for finite time horizons $T = 1$ and $T = 10$. Also, plot the gains gains as a function of time.

part (d): (3 points) Compare the infinite horizon and finte horizon optimal control solutions. Which finite time solution is the closest to the infinite time solution? How do the gains differ?

Problem #2: (15 points) Consider the following optimal control problem. You are given a linear dynamical system

$$\dot{x} \;=\; Ax \;+\; Bu \tag{5}$$

and wish to design the optimal control $u(t)$ which optimizes the following performance index:

$$J(x,u) \;=\; \frac{1}{2}\int_0^T \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & V \\ V^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \;+\frac{1}{2}X^T(T)P_T x(T)\;. \tag{6}$$

If we define the *modified Kalman gain* as

$$K = R^{-1}(V^T + B^T P) \tag{7}$$

then show that the optimal control for this problem is given by:

$$u(t) \;=\; -K(t)x(t) \tag{8}$$

where P is the solution to the following Riccati-type equation

$$-\dot{P} \;=\; PA \;+\; A^T P \;-\; K^T R K \;+Q \tag{9}$$

with terminal condition $P(T) = P_T$. You can assume that Q, R, and P are symmetric.