CDS 112: Winter 2014/2015

Homework #3: due Wednesday, Feb. 11, 2015

Problem #1: (15 points) In this problem, you will show (using two different methods) that the shortest path between two points in the plane is a straight line. Let $\vec{z}_0 = [x_0 \; y_0]^T$ and $\vec{z}_f = [x_f \; y_f]^T$ be two points in $\mathbb{R}^2$ which are to be connected by a path.

- **Part (a):** Use the Pontryagin Minimum Principle to find the control for the system

  $$\dot{\vec{z}} \;=\; \vec{u}$$

  which minimizes the cost

  $$J \;=\; \int_0^1 ||\dot{\vec{z}}|| \; dt \;=\; \int_0^1 \sqrt{\dot{\vec{z}}^T \dot{\vec{z}}} \; dt$$

  subject to the conditions that $\vec{z}(t = 0) = \vec{z}_0$ and $\vec{z}(t = 1) = \vec{z}_f$. There are two control inputs, $\vec{u} = [u_x \; u_y]^T$.

- **Part (b):** Use the Calculus of Variations to find the trajectory $y(x)$ which minimizes the cost

  $$J \;=\; \int_0^1 \sqrt{1 + \dot{y}^2} \; dx$$

  subject to the conditions that $y(x = 0) = 0$ and $y(x = 1) = 1$.

In both cases, show that the resulting system travels a straight line.

Problem #2: (15 points) Derive the LQR solution for a finite-horizon discrete-time optimal control problem using Bellman's equation. That is, given the time-varying value cost:

$$V(x, k) \;=\; \sum_{i=k}^{N-1} (x_i^T Q x_i \;+\; u_i^T R u_i) \;+\; x_N^T Q_N x_N \tag{1}$$

derive the feedback law $u_k \;=\; K_k x_k$ that minimizes this cost. Start with Bellman's equation, adapted to the discrete time case:

$$V(x_k, k) \;=\; \min_u \left[L(x, k) \;+\; V(x_{k+1}, k + 1)\right]$$

and assume that $V(x_k, k) \;=\; x_k^T P_k x_k$.

Problem #3: (10 points) In this problem you will use the Hamilton-Jacobi-Bellman equation to design a controller for the nonlinear system

$$\dot{x} \;=\; -x^3 \;+\; u$$

where $x \in \mathbb{R}$ is the system state, and u is the control. This is a simplified model of a mechanical oscillator with a *hardening spring*. Design the control to minimize the cost function:

$$J = \frac{1}{2} \int_0^{\infty} (x^2 + u^2)\ dt\ .$$

Problem #4: (20 points) This problem will extend Problem 1 of Homework set #2. Recall that you are given a system whose dynamics are governed by the equations:

$$\dot{x} = ax + bu \tag{2}$$

where $x \in \mathbb{R}$ denotes the state, $u \in \mathbb{R}$ is a single scalar control input, and a, b are constant, *positive* scalars:

$$a = 2 \qquad b = 0.5\ . \tag{3}$$

In the previous homework, you designed a finite horizon optimal optimal controller to minimize the cost

$$J = \frac{1}{2} \int_{t_0}^{T} u^2(t)\ dt + \frac{1}{2} c\ x^2(T) \tag{4}$$

where $c > 0$ is a constant (taking the values of either $c = 0.1$ or $c = 10$).

In this problem, you are to implement the finite horizon controller in a *receding horizon* fashion and compare it against a longer duration finite horizon controller. That is, for $x(0) = 4$, implement the finite horizon controller with $T = 0.5$, but updated every 0.5 time units over a total horizon of 10 time units. Plot the trajectory of the controlled system over the 10 unit time horizon. Do this for both $c = 0.1$ and $c = 10$, as in the previous homework. Then, compare your results against the previous homework where you implemented the finite horizon controller for $T = 10$ time units.