# CDS 112: Winter 2014/2015

Homework #5: due Wednesday, March 4, 2015

**Problem #2:** (35 Points) In this problem, we try to gain some intuition about how estimators work in general by learning about and applying one of the simplest, yet surprisingly effectively, techniques known as **Least Squares Estimation.**

Let the quantity $y \in \mathbb{R}$ be related linearly to variable $x \in \mathbb{R}^n$ via the following relationship:

$$y = \theta^T x = \sum_{i=1}^{n} \theta_i x_i$$

Suppose that you have $N$ noisy measurements of the form $(x_i, y_i)$, and you want to estimate $\theta$. In general, because the measurements aren't perfect, there may not exist a $\theta$ that will satisfy

$$y_i = \theta^T x_i \qquad i = 1, 2, \dots N$$

We can, however, try make an educated guess $\hat{\theta}$ that might be close to $\theta$. One way to do this is to assign to each $\theta \in \mathbb{R}^n$ a cost $J(\theta)$, and pick $\hat{\theta}$ as the minimizer of this cost. A particular cost that is very widely used is

$$J(\theta) = \frac{1}{2} \sum_{i=1}^{N} (y_i - \theta^T x_i)$$

in words, this cost is the *sum of squared errors.*

- **Part (a):** Let

$$A \triangleq \begin{bmatrix} - x_1^T - \\ - x_2^T - \\ \vdots \\ - x_N^T - \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

We are hoping to find $\hat{\theta}$ that is the *closest* to satisfying

$$A\hat{\theta} = Y$$

We can rewrite $J(\theta)$ from above as

$$J(\theta) = \|Y - A\theta\|_2^2 \ .$$

**show that**

$$\theta^* = \arg\min_{\theta} J(\theta) = (A^T A)^{-1} A^T Y$$

**Hint:** Use the fact that $\|x\|_2^2 = x^T x$.
Note: We call $\theta^*$ the *'Least-Squares solution'* to the equation

$$A\theta = Y$$

which `MATLAB` will give you if you type

$$A\backslash Y$$

This solution applies whenever $A$ is a full rank, tall/skinny matrix.

- **Part (b):** Suppose that we don't get measurements all at once, and instead we get them sequentially. Let $\hat{\theta}_N$ be the least-squares solution obtained front the first $N$ measurements. We obtain a new measurement, and we want to use this information to compute $\hat{\theta}_{N+1}$ without recomputing the solution from scratch. We can do this by using the 'Recursive Least Squares Filter'.

  Let

$$P_N = A_N^T A_N \qquad \text{where} \quad A_N = \begin{bmatrix} — x_1^T — \\ — x_2^T — \\ \vdots \\ — x_N^T — \end{bmatrix}$$

  Show that the recursion

$$\hat{\theta}_N = \hat{\theta}_{N-1} + P_N^{-1} x_N (y_N - x_N^T \hat{\theta}_{N-1})$$

  with

$$P_N^{-1} = P_{N-1}^{-1} - P_{N-1}^{-1} x_N (1 + x_N^T P_{N-1}^{-1} x_N)^{-1} x_N^T P_{N-1}^{-1}$$

  produces the least squares solution using available measurements. That is, show that the formula from part 1 and this recursion agree for any N.

  **Hints:** To show that the formula for $\hat{\theta}_N$ holds, use the least squares solution from part 2 (you might want to rewrite it it in terms of $P_N$ and rearrange), and the fact that

$$P_N = P_{N-1} + x_N x_N^T$$

  To show that the Recursion for $P_N^{-1}$ holds, use

  **The Matrix Inversion Lemma**
  For matrices, $A, B, C, D$ of appropriate dimensions with $A$ invertible, The following holds

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)DA^{-1}$$

- **Part (c):** It is 3050 A.D. and NASA has managed to send a spacecraft light-years away, towards ahabitable exoplanet. The spacecraft enters the planet's atmosphere, and the astronauts on board realize that they have no idea what the atmospheric drag coefficient ($c_{atm}$) or the acceleration due to gravity ($g$) of this new planet is. This would help them plan a safe and smooth descent trajectory to the surface. The spacecraft is known to fall according to

$$m_s \ddot{z} - c_{atm} |\dot{z}| = -m_s g$$

  where $z(t)$ is the spacecraft's altitude.

1. In the first scenario, the people on board never solved part (b) above, and so they crash-land (don't worry, everyone survives). They collected measurements of $\ddot{z}$ and $|\dot{z}|$ throughout the fall, and their data is available as `data.m` (the first column is $\ddot{z}$ and the second column is $|\dot{z}|$.) Obtain least-squares estimates for $g$ and $c_{atm}/m_s$ using the result from part 1.

   **Hint:** *Pick one of the measured values to be $y$, and the other to be part of $x$ (you need to add something to the end of each $x$ to account for a constant offset).*

2. In the second scenario, the people on board solved part 2, and they decide to use the RLS filter – Implement the recursion from part 2 (set $P_0 = \mathbb{I}_{n\times n} \times 10^6$ and $\hat{\theta}_0 = \vec{0}$). Plot your estimates of $g$ and $c_{atm}/m_s$ as a function of $N$, where $N$ ranges from 1 to number of measurements. Do your estimates converge to the answers from the previous question?

**Problem #2:** (25 points)

In the Kidnapped robot problem, a robot is "blindfolded" and taken to a new location. Once the blindfold is removed, the robot must localize (determine its position relative to some known coordinate system) itself as quickly as possible. In the classical problem, the robot only has a map and its on-board sensors (e.g., visual sensors to measure the range and distance to nearby objects in its environment) to determine its location. In this CDS 112 version of the problem, the robot has an on-board GPS system as well as an Inertial Measurement Unit (IMU). IMUs conventionally provide (noisy) measurements of the robot's acceleration, as well as gyroscopes to provide (noisy) measurements of the robot's rate of rotation. The IMU measurements can be fused with those of a GPS receiver to provide high rate updates on the robot's position between GPS measurements, and to smooth out GPS measurement errors. During GPS blackouts the IMU provides the only position reference, which may be essential for the robot's navigation and operation.

In this highly simplified version of the problem, we will assume that the robot is restricted to move along the x-axis, and the accelerometer measures acceleration along that axis. Let x denote the vehicle position. The dynamics governing the robot's motion are:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} = u \tag{1}$$

where $m$ is the vehicle mass, $b$ is the vehicle damping coefficient (e.g., to account for wind resistance or drag on the robot wheels), and $u$ is the control input (in units of force) which governs the vehicle motion. For this problem, let $m = 50\ kg$ and let $b/m = 0.01s^{-1}$.

- **Part (a):** develop a discrete time dynamical model of this system's equations of motion.

- **Part (b):** Assume the IMU can measure vehicle acceleration at a rate of 20 Hz, with an accuracy of $0.0981m/sec^2$ (which is 1% of gravity). Assume that GPS measurements are received at a rate of 2 Hz, with the measurement have a standard deviation of 5.0 meters. Assume that both measurement uncertainties can be modeled as zero mean Gaussian distributed white noise with std. deviation equal to the stated accuracies. Assume that when the robot's blindfold is taken off, its initial position uncertainty can be modeled by a white Gaussian process with variance of 100 meters. Assuming there are no external disturbance forces acting on the vehicle, compute the time history of the estimation covariance, and plot the covariance associated with the positional portion of the covariance for the first 10 seconds of vehicle motion. Assume that at the beginning of the motion, a GPS measurement of vehicle position is not available, but an IMU measurement of vehicle acceleration is available.