CDS 112 Lecture 1
Course Overview/Organization

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Goals for Today:

- Course administration for CDS 110b
- Introduce modern (optimization-based) control system
- Introduction to optimization
- Conflicts?

Course Admin

Lectures:

- MWF 1:00-1:55; Annenberg 107 (today). Then Annenberg 213 thereafter

Web page:

- https://www.cds.caltech.edu/~murray/wiki/index.php/CDS_112,Winter_2015
- Copies of any Notes handed out in class, Copies of lecture slides
- Course text(s)
- Reading assignments. Homeworks and their solutions (see Schedule page)
- Course announcements, etc.

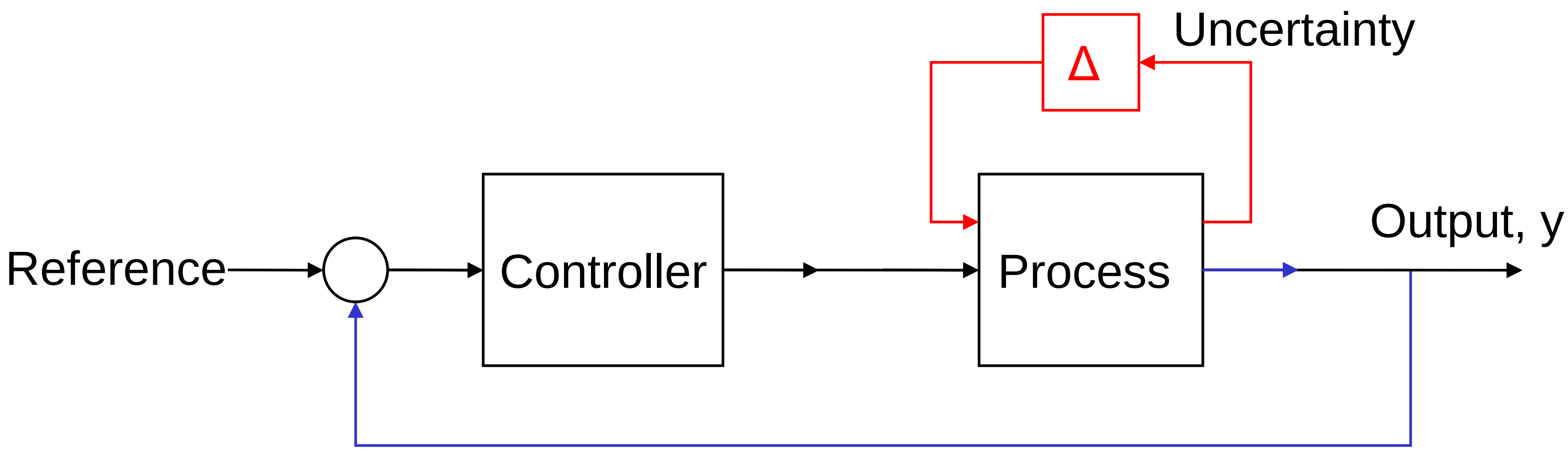
Course texts: (all free)

- R. M. Murray, *Optimization-based Control*, 2010 Notes/Preprint (link on website)
- B.D.O. Anderson & J.B. Moore, *Optimal Filtering* (link on website)
- Others:
  - Doyle, Francis, Tannenbaum, *Feedback Control Theory*, (link on website)
  - Friedland, *Control System Design,* (in SFL)
  - Other reading: see website

Grading:

- 70% Homework: ~6 homeworks every ~10 days. Due at 5:00 pm (box outside 205 Keck)
  ***Two* 2-day grace periods can be used any time during the quarter**
- 30% Final: “open-book,” covering weeks 1-9

Control System Design: 110



Design controller

- Use process model, design in frequency domain (Bode, Nyquist)
- Use process model, design in state-space with state feedback (pole placement)

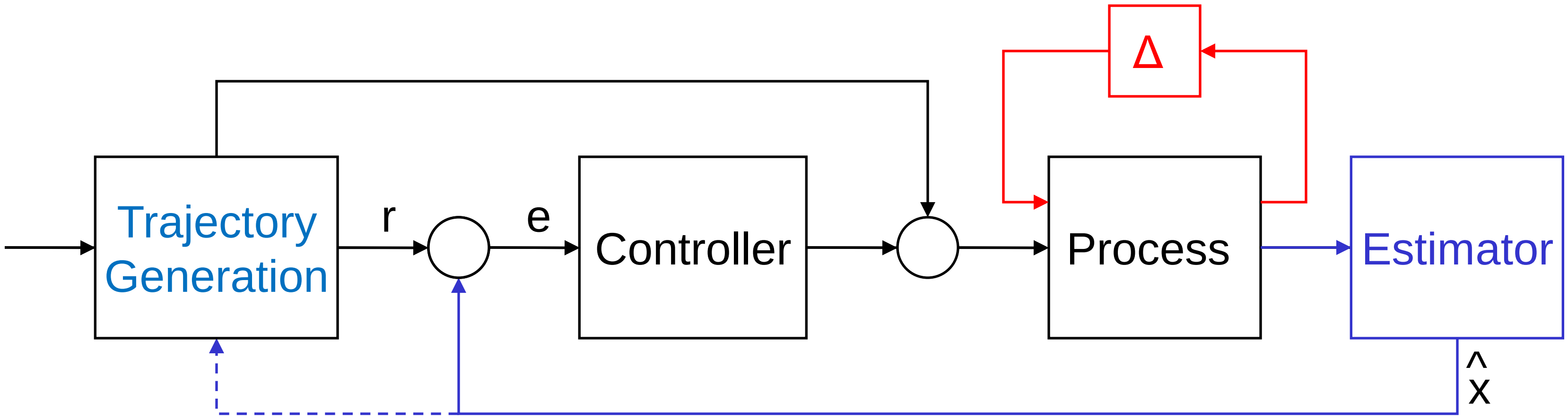
Robustness to uncertainty

- Intuitive, through phase & stability margins

Goals:

- Stabilization to a point (maybe unstable equilibrium)
- Tracking (follow simple reference trajectory, such as step input)
- Disturbance rejection (maintain equilibrium despite disturbances)

Control System Design: 112



Design controller

- Use process model, design in state-space with ***estimator*** and state feedback
- Generalization of CDS 110 ***observer***

Robustness to uncertainty

- Formally, with rigorous analysis (CDS 212-213)

Goals:

- Stabilization (maybe unstable equilibrium)
- Disturbance rejection (maintain equilibrium despite disturbances)
  – *Modeled as random variables in CDS 112*
  – *Modeled as sets in CDS 212.*
- Trajectory generation: Include design of reference as part of control design to accomplish some higher-level task
- Tracking (follow unknown reference trajectory, possibly with feedforward)

Classical (110) vs “Modern” (112, 212, 213 …) Control Design

Classical: (1940’s to ~1960)

- Frequency domain design
- Graphical tools (Bode, Nyquist, Nichols, Root Locus)
- Intuition about how to tweak design
- MIMO hard!

“Modern”: (1960’s to…)

- State-space (time-domain)
- MIMO handled automatically
- Extends to nonlinear more easily
- Systematic design procedure
- Optimal design (really starting in 1950’s)

- Key ideas:
  - Analytical specification of control goals
  - Systematic design for minimum “cost” (some performance metric)

Classical vs "Modern" Control Design

Plant model:

$$\begin{aligned}\dot{x} &= f(x,u)\\ y &= g(x)\end{aligned}$$

Linearize:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

Transfer function

$$\begin{aligned}y(s) &= C(sI-A)^{-1}Bu(s)\\ &= G(s)u(s)\end{aligned}$$

Design

$$u(s) = -K(s)y(s)$$

Design estimator

$$\dot{\hat{x}} = A\hat{x} + L(y - C\hat{x}) + Bu$$

Optimal: CDS 112
Robust: CDS 212

Design state fdbk

$$u = -K(\hat{x} - x_d) + u_d$$

Implement:

$$\begin{aligned}\dot{x}_c &= A_c x_c + B_c(y-r)\\ u &= C_c x_c + k_r r\end{aligned}$$

Course Overview

1. Optimization:

Given $J(x,u,t,T) = \int_0^T l(x,u,t)\,dt + V\big(x(T),u(T)\big)$

$$\dot{x} = f(x,u,n)$$

Find $u = u(x,t,T) = \arg\min_u J$

- Optimal state feedback (linear sys. u = –Kx)
- On-line implementation of optimal control (RHC)
- Trajectory generation (calculus of variations)

Weeks 1 – 4

2. Optimal estimation:

Given $\dot{x} = f(x,u,n)$ y= $h(x) + \omega$

Find ***best*** estimate of x: $\hat{x}$

- Optimization: minimize covariance of estimate error…
- need to model stochastic processes

Weeks 5 – 9

3. Robustness
   - How to quantify, analyze, and design for uncertainty
   - Formalizes classical control loop-shaping ideas
   - Connects time- and frequency-domain design

Week 10 ? (CDS 212)

4. More focus on discrete-time control

Optimal Control of Systems

(**Assignment:** start Reading Chap. 2 of Optimal Control Notes)

Given a system:

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n,\ u \in \Omega \subset \mathbb{R}^p$$

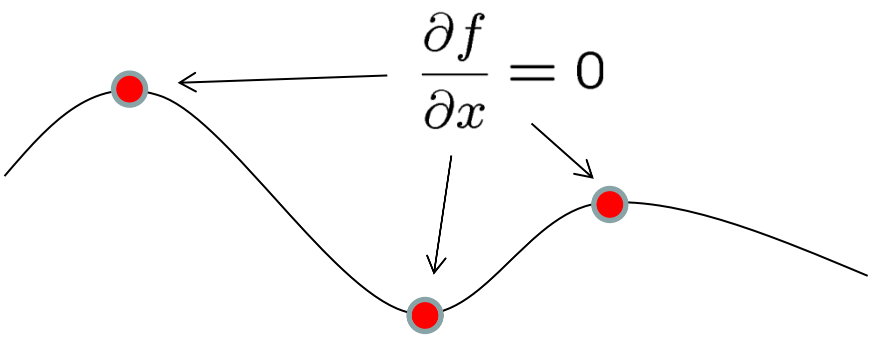
with $x(0) = x_0$. Then find

$$u = \mathrm{argmin}_{u \in \Omega} \left(\underbrace{\int_0^T \overbrace{L(x,u)}^{\textit{Instantaneous, (Stage) Cost}} \mathrm{d}t + \overbrace{V(x(T), u(T))}^{\textit{Terminal Cost}}}_{\text{J(x,u)}} \right)$$

- Can include additional constraints on control *u*, and on state (along trajectory or at final time):
- Final time *T* may or may not be free (we'll first derive fixed *T* case)
- Define $z = \begin{bmatrix} x \\ u \end{bmatrix}$, then this is a problem of minimizing J(z) subject to constraints G(z)=0

Function Optimization

- Necessary condition for optimality is that gradient is zero
  - Characterizes local extrema; need to check sign of second derivative
  - Need convexity to guarantee global optimum



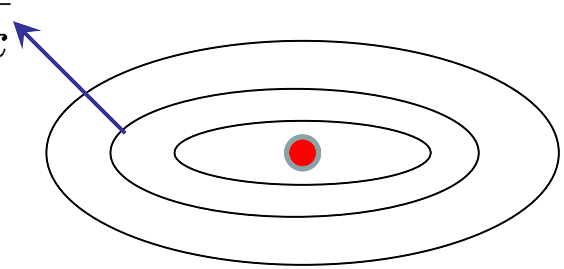
Keep this analogy in mind:
at an extremum

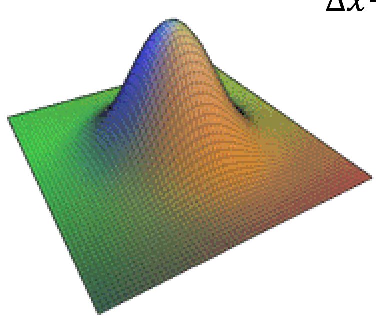
$$\lim_{\Delta x \to 0} \{ f(x^* + \Delta x) - f(x^*) \} = 0$$

$$\lim_{\Delta x \to 0} \left\{ f(x^*) + \frac{\partial f(x^*)}{\partial x} \Delta x + \ \ldots . - f(x^*) \right\} = 0$$

Gradient of f(x)
orthogonal to level sets

$\frac{\partial f}{\partial x}$

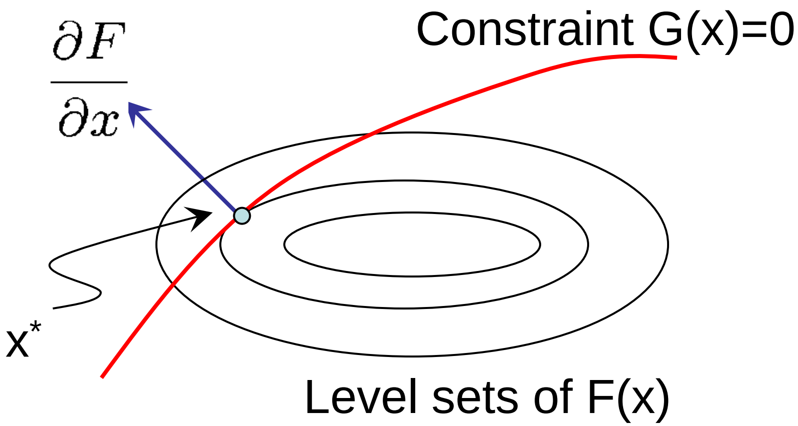




Level sets of f(x)

Constrained Function optimization

Given $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $G_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1 \ldots k$, then find $x^* \in \mathbb{R}^n$ such that $G_i(x^*) = 0 \, \forall i$ and $F(x^*) \geq F(x)$ for all x satisfying $G_i(x) = 0 \, \forall i$.



- Then at optimal solution, gradient of F(x) must be parallel to gradient of G(x):

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0$$

- More generally, define:

Lagrange multiplier

$$\tilde{F} = F + \lambda^T G$$

- Then a necessary condition is:

$$\frac{\partial \tilde{F}}{\partial \tilde{x}}(\tilde{x}^*) = 0 \qquad \tilde{x} = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

- The *Lagrange multipliers* λ are the sensitivity of the cost to a change in G

Optimal Control of Systems

Given a system:

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n, \boxed{u \in \Omega \subset \mathbb{R}^p}$$

with $x(0) = x_0$. Then find

$$u = \operatorname{argmin}_{u \in \Omega} \left(\underbrace{\int_0^T L(x,u)\mathrm{d}t + V(x(T), u(T))}_{\text{J(x,u)}} \right)$$

- Easy to include additional constraints on control *u*, and on state (along trajectory or at final time)
- Final time *T* may or may not be free (I'll only derive for fixed *T*)
- Define $z = \begin{bmatrix} x \\ u \end{bmatrix}$, then this is a problem of minimizing J(z) subject to constraints G(z)=0

Solution approach

- Add Lagrange multiplier λ(t) for dynamic constraint
  - And additional multipliers for terminal constraints or state constraints

- Form augmented cost functional:

$$\begin{aligned}\tilde{J}(x,u,\lambda) &= J(x,u) + \int_0^T \lambda^T \left(f(x,u) - \dot{x}\right) \mathrm{d}t \\ &= \int_0^T \Big(L(x,u) + \lambda^T (f(x,u) - \dot{x})\Big)\, \mathrm{d}t + V(x(T)) \\ &= \int_0^T \Big(H(x,u,\lambda) - \lambda^T \dot{x}\Big)\, \mathrm{d}t + V(x(T))\end{aligned}$$

- where the ***Hamiltonian*** is: $H \triangleq L + \lambda^T f$

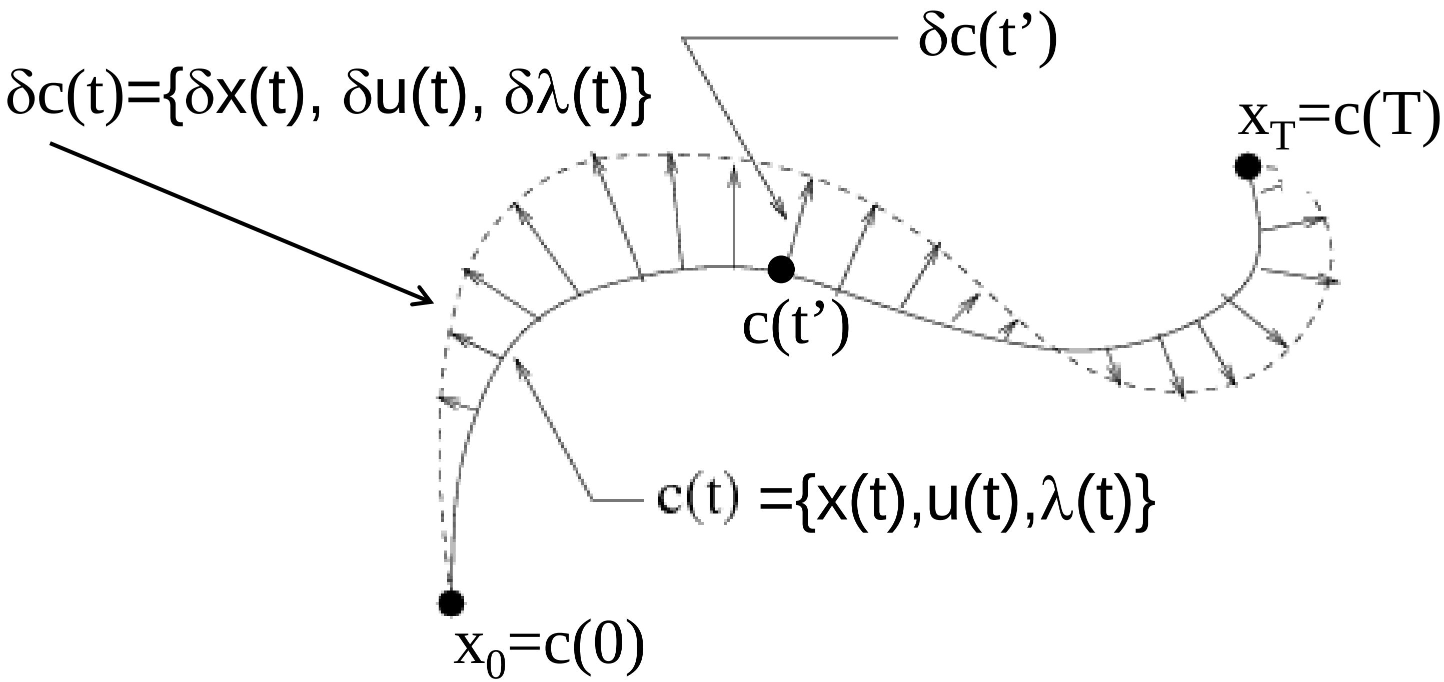
- Necessary condition for optimality: $\delta \tilde{J}$ vanishes for any perturbation (variation) in x, u, or λ about optimum:

"variations"

$$\left.\begin{aligned} x(t) = x^*(t) + \delta x(t); \quad u(t) = u^*(t) + \delta u(t); \quad \lambda(t) = \lambda^*(t) + \delta\lambda\,(t); \end{aligned}\right\}$$

Variations must satisfy path end conditions!

variation



Derivation…

$$\delta\tilde{J} = \tilde{J}\,(x^* + \delta x, u^* + \delta u, \lambda^* + \delta\lambda) \;-\; \tilde{J}(x^*, u^*, \lambda^*)$$

$$\begin{aligned}\delta\tilde{J} &\triangleq \tilde{J} - \tilde{J}^* \\ &\simeq \int_0^T \left(\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u - \lambda^T\delta\dot{x} + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda\right)\mathrm{d}t + \frac{\partial V}{\partial x}\delta x(T)\end{aligned}$$

- Note that (integration by parts):

$$\int_0^T \lambda^T\delta\dot{x} = -\int_0^T \dot{\lambda}^T\delta x + \lambda^T(T)\delta x(T) - \lambda^T(0)\delta x(0)$$

- So:

$$\begin{aligned}\delta\tilde{J} = \int_0^T &\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)\delta\lambda\right]\mathrm{d}t \\ &+ \left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)\delta x(T) + \lambda^T(0)\delta x(0)\end{aligned}$$

We want this to be ***stationary*** for all variations

Pontryagin's Maximum Principle

- Optimal (x^*,u^*) satisfy:

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

$$H(x^*(t), u^*(t), \lambda^*(t)) \leq \boxed{H(x^*(t), u, \lambda^*(t)) \;\forall u \in \Omega}$$

Optimal control is solution to O.D.E.

- If $\Omega = \mathbb{R}^m$ and H differentiable then $\partial H / \partial u = 0$
- Can be more general and include terminal constraints
- Follows directly from:

$$\delta \tilde{J} = \int_0^T \left[\underbrace{\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)}_{0} \delta x + \frac{\partial H}{\partial u} \delta u + \underbrace{\left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)}_{0} \delta \lambda \right] \mathrm{d}t$$

$$+ \underbrace{\left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)}_{0} \delta x(T) + \lambda^T(0) \underbrace{\delta x(0)}_{0}$$

Interpretation of λ

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0 \qquad \leftarrow \quad \dot{x} = f(x,u)$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

- Two-point boundary value problem: λ is solved backwards in time
- λ is the “co-state” (or “adjoint” variable)
- Recall that $H = L(x,u) + \lambda^T f(x,u)$
- If L=0, $\lambda(t)$ is the sensitivity of the cost to a perturbation in state x(t)
  - In the integral as $\lambda(t)\delta\dot{x}$
  - Recall $\delta J = \ldots + \lambda(0)\delta x(0)$

$$x(\tau^+) = x(\tau^-) + \epsilon$$
$$\Rightarrow \delta\dot{x} = \epsilon\delta_D(t-\tau)$$
$$\Rightarrow \delta\tilde{J} = \int \cdots \lambda^T \delta\dot{x} \cdots = \lambda^T(\tau)\epsilon$$