Optimal Control of Systems

Given a system: $\dot{x} = f(x,u,n)$; $x \in \mathbb{R}^n$, $u \in \Omega \subset \mathbb{R}^p$; $x(0) = x_0$

Find the control u(t) for $t \,\varepsilon\, [0,\mathrm{T}]$ such that

Instantaneous (Stage) Cost *Terminal Cost*

$$u = \arg \lim_{u \epsilon \boldsymbol{\Omega}} \underbrace{\int_0^T \overbrace{L(x,u)}^{\text{Instantaneous (Stage) Cost}} dt + \overbrace{V\big(x(T), u(T)\big)}^{\text{Terminal Cost}}}_{J(x,u)}$$

- Can include constraints on control u and state x
  - (along trajectory or at final time):
- Final time *T* may or may not be free (we'll first derive fixed *T* case)

Solution approach

- Add Lagrange multiplier λ(t) for dynamic constraint
  - And additional multipliers for terminal constraints or state constraints
- Form augmented cost functional:

$$\begin{aligned}\tilde{J}(x,u,\lambda) &= J(x,u) + \int_0^T \lambda^T \left(f(x,u) - \dot{x}\right) \mathrm{d}t \\ &= \int_0^T \Big(L(x,u) + \lambda^T(f(x,u) - \dot{x})\Big)\, \mathrm{d}t + V(x(T)) \\ &= \int_0^T \Big(H(x,u,\lambda) - \lambda^T \dot{x}\Big)\, \mathrm{d}t + V(x(T))\end{aligned}$$

- where the ***Hamiltonian*** is: $H \triangleq L + \lambda^T f$
- Necessary condition for optimality: $\delta\tilde{J}$ vanishes for any perturbation (variation) in x, u, or λ about optimum:

"variations"

$$\left.\begin{aligned} x(t) = x^*(t) + \underline{\delta x(t)}; \quad u(t) = u^*(t) + \underline{\delta u(t)}; \quad \lambda(t) = \lambda^*(t) + \underline{\delta\lambda\,(t)}; \end{aligned}\right\}$$

Variations must satisfy path end conditions!

Pontryagin's Maximum Principle

- Optimal (x*,u*) satisfy:

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

$$H(x^*(t), u^*(t), \lambda^*(t)) \leq \boxed{H(x^*(t), u, \lambda^*(t)) \ \forall u \in \Omega}$$

Optimal control is solution to O.D.E.

- If $\Omega = \mathbb{R}^m$ and H differentiable then $\partial H/\partial u = 0$
- Can be more general and include terminal constraints
- Follows directly from:

$$\delta \tilde{J} = \int_0^T \left[\underbrace{\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)}_{0} \delta x + \frac{\partial H}{\partial u}\delta u + \underbrace{\left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)}_{0} \delta \lambda\right] \mathrm{d}t$$
$$+ \underbrace{\left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)}_{0} \delta x(T) + \lambda^T(0)\underbrace{\delta x(0)}_{0}$$

Unbounded controls

Interpretation of λ

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0 \qquad \leftarrow \quad \dot{x} = f(x,u)$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

- Two-point boundary value problem: λ is solved backwards in time
- λ is the "co-state" (or "adjoint" variable)
- Recall that $H = L(x,u) + \lambda^T f(x,u)$
- If L=0, λ(t) is the sensitivity of the cost to a perturbation in state x(t)
  - In the integral as $\lambda(t)\delta\dot{x}$
  - Recall δJ = … +λ(0)δx(0)

$$x(\tau^+) = x(\tau^-) + \epsilon$$
$$\Rightarrow \delta\dot{x} = \epsilon\delta_D(t-\tau)$$
$$\Rightarrow \delta\tilde{J} = \int \cdots \lambda^T \delta\dot{x} \cdots = \lambda^T(\tau)\epsilon$$

Terminal Constraints

Assume ***q*** terminal constraints of the form: $\psi(x(T))=0$

- Then

$$\lambda(T) = \left(\frac{\partial V}{\partial x}\right)\left(x(T)\right) + \left(\frac{\partial \Psi}{\partial x}\right)(x(T))\, \nu$$

- Where ν is a set of undetermined Lagrange Multipliers

- Under some conditions, ν is free, and therefore $\lambda(T)$ is free as well

When the final time T is free (i.e., it is not predetermined), then the cost function *J* must be stationary with respect to perturbations T: $T^* + \delta T$. In this case:

$$H(T) = 0$$

Example: Bang-Bang Control

Consider time optimal control of linear system

- $\dot{x} = Ax + Bu$
- $x(0) = x_0$; $x(T) = x_F$; $\varphi(x(T)) = x_F - x(T)$
- $|u| \leq 1$; $J = \int_0^T 1\, dt$ "minimum time control"

Apply PMP:

- $H = L + \lambda^T f = 1 + \lambda^T (Ax + Bu) = 1 + (\lambda^T A)x + (\lambda^T B)u$
- $\dot{x} = \left(\frac{\partial H}{\partial x}\right)^T = Ax + Bu$
- $u = \operatorname{argmin}(H) = -\operatorname{sgn}(\lambda^T B)$
- I.e., control u is +1 or -1 in value

Since H is linear w.r.t. u,
minimization occurs at boundary