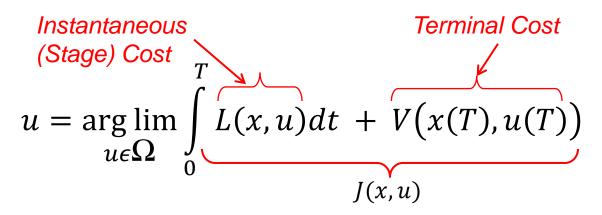
Optimal Control of Systems

Given a system: $\dot{x} = f(x, u, n)$; $x \in \mathbb{R}^n$, $u \in \Omega \subset \mathbb{R}^p$; $x(0) = x_0$

Find the control u(t) for $t \in [0,T]$ such that



- Can include constraints on control u and state x
 - (along trajectory or at final time):
- Final time T may or may not be free (we'll first derive fixed T case)

Solution approach

- Add Lagrange multiplier λ(t) for dynamic constraint
 - And additional multipliers for terminal constraints or state constraints
- Form augmented cost functional:

$$\tilde{J}(x, u, \lambda) = J(x, u) + \int_0^T \lambda^T (f(x, u) - \dot{x}) dt$$

$$= \int_0^T \left(L(x, u) + \lambda^T (f(x, u) - \dot{x}) \right) dt + V(x(T))$$

$$= \int_0^T \left(H(x, u, \lambda) - \lambda^T \dot{x} \right) dt + V(x(T))$$

- where the *Hamiltonian* is: $H \triangleq L + \lambda^T f$
- Necessary condition for optimality: $\delta \tilde{J}$ vanishes for any perturbation (variation) in x, u, or λ about optimum:

"variations"

$$x(t) = x^*(t) + \underline{\delta x(t)}; \qquad u(t) = u^*(t) + \underline{\delta u(t)}; \qquad \lambda(t) = \lambda^*(t) + \underline{\delta \lambda}(t);$$

Variations must satisfy path end conditions!

Pontryagin's Maximum Principle

Optimal (x*,u*) satisfy:

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^{T} \qquad x(0) = x_{0}$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^{T} \qquad \lambda(T) = \left(\frac{\partial V}{\partial x}\Big|_{x=x(T)}\right)^{T}$$

Optimal control is solution to O.D.E.

$$H(x^*(t), u^*(t), \lambda^*(t)) \le H(x^*(t), u, \lambda^*(t)) \ \forall u \in \Omega$$

Unbounded controls

• Can be more general and include terminal constraints

If $\Omega = \mathbb{R}^m$ and H differentiable then $\partial H/\partial u = 0$

• Follows directly from: 0 $\delta \tilde{J} = \int_0^T \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T \right) \delta x + \frac{\partial H}{\partial u} \delta u + \left(\frac{\partial H}{\partial \lambda} - \dot{x}^T \right) \delta \lambda \right] dt$ $+ \left(\frac{\partial V}{\partial x} - \lambda^T (T) \right) \delta x (T) + \lambda^T (0) \delta x (0)$

Interpretation of λ

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^{T} \qquad x(0) = x_{0} \qquad \leftarrow \dot{x} = f(x, u)$$
$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^{T} \qquad \lambda(T) = \left(\frac{\partial V}{\partial x}\Big|_{x=x(T)}\right)^{T}$$

- Two-point boundary value problem: λ is solved backwards in time
- λ is the "co-state" (or "adjoint" variable)
- Recall that $H = L(x,u) + \lambda^T f(x,u)$
- If L=0, λ(t) is the sensitivity of the cost to a perturbation in state x(t)
 - In the integral as $\lambda(t)\delta\dot{x}$
 - Recall $\delta J = \dots + \lambda(0)\delta x(0)$

$$x(\tau^{+}) = x(\tau^{-}) + \epsilon$$

$$\Rightarrow \delta \dot{x} = \epsilon \delta_{D}(t - \tau)$$

$$\Rightarrow \delta \tilde{J} = \int \cdots \lambda^{T} \delta \dot{x} \cdots = \lambda^{T}(\tau) \epsilon$$

Terminal Constraints

Assume q terminal constraints of the form: $\psi(x(T))=0$

• Then

$$\lambda(T) = \left(\frac{\partial V}{\partial x}\right) \left(x(T)\right) + \left(\frac{\partial \Psi}{\partial x}\right) \left(x(T)\right) V$$

- Where v is a set of undetermined Lagrange Multipliers
- Under some conditions, v is free, and therefore $\lambda(T)$ is free as well

When the final time T is free (i.e., it is not predetermined), then the cost function J must be stationary with respect to perturbations T: T* + δ T. In this case:

$$H(T) = 0$$

Example: Bang-Bang Control

Consider time optimal control of linear system

- $-\dot{x} = Ax + Bu$
- $x(0) = x_0; x(T) = x_F; \varphi(x(T)) = x_F x(T)$
- $|u| \le 1;$ $J = \int_0^T 1 dt$

 "minimum time control"

Apply PMP:

•
$$H = L + \lambda^T f = 1 + \lambda^T (Ax + Bu) = 1 + (\lambda^T A)x + (\lambda^T B)u$$

•
$$\dot{x} = \left(\frac{\partial H}{\partial x}\right)^T = Ax + Bu$$

- $u=argmin(H) = -sgn(\lambda^T B)$
- I.e., control u is +1 or -1 in value

Since H is linear w.r.t. u, — - - minimization occurs at boundary

