Review of PMP Derivation

We want to find control u(t) which minimizes the function:

$$J(x,u) = \int_0^T L(x,u)dt \; + \; V\big(x(T),u(T)\big)$$

Subject to the constraints: $x(t=0) = x_0,$ T fixed, and $\dot{x} = f(x,u)$

To find a necessary condition to minimize $J(x,u)$, we *augment* the cost function

$$\begin{aligned}\tilde{J}(x,u,\lambda) &= J(x,u) + \int_0^T \lambda^T \left(f(x,u) - \dot{x}\right) \mathrm{d}t \\ &= \int_0^T \left(L(x,u) + \lambda^T(f(x,u) - \dot{x})\right) \mathrm{d}t + V(x(T)) \\ &= \int_0^T \left(H(x,u,\lambda) - \lambda^T \dot{x}\right) \mathrm{d}t + V(x(T))\end{aligned}$$

Then *extremize* the augmented cost with respect to the variations

$x(t) = x^*(t) + \delta x(t)$; $\quad u(t) = u^*(t) + \delta u(t)$; $\quad \lambda(t) = \lambda^*(t) + \delta\lambda(t)$;

Pontryagin's Maximum Principle

- Optimal (x^*,u^*) satisfy:

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T \qquad x(0) = x_0$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \left(\left.\frac{\partial V}{\partial x}\right|_{x=x(T)}\right)^T$$

$$H(x^*(t), u^*(t), \lambda^*(t)) \leq H(x^*(t), u, \lambda^*(t)) \ \forall u \in \Omega$$

Optimal control is solution to O.D.E.

- If $\Omega = \mathbb{R}^m$ and H differentiable then $\partial H / \partial u = 0$

Unbounded controls

- Can be more general and include terminal constraints
- Follows directly from:

$$\delta \tilde{J} = \int_0^T \left[\underbrace{\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)}_{0} \delta x + \frac{\partial H}{\partial u} \delta u + \underbrace{\left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)}_{0} \delta \lambda \right] \mathrm{d}t + \underbrace{\left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)}_{0} \delta x(T) + \lambda^T(0) \underbrace{\delta x(0)}_{0}$$

Handling Additional Constraints

Example: what if *final state* constraints $\phi_i(x(T))=0$, $i=1,\ldots,p$ are desired:

– Add constraints to cost using additional Lagrange multipliers

Multipliers

$$J_a(x,u) = \int_0^T L(x,u)dt + V\big(x(T),u(T)\big) + \underbrace{\nu_1\varphi_1\big(x(T)\big) + \ldots + \nu_p\varphi_p\big(x(T)\big)}_{\text{Cost augmentation}}$$

To find a necessary condition for optimal, extremize the constrained, augmented the cost function

$$\widetilde{J_a}(x,u) = \int_0^T (H(x,u,\lambda) - \lambda^T \dot{x})dt + V\big(x(T),u(T)\big) + \nu_1\varphi_1\big(x(T)\big) + \ldots + \nu_p\varphi_p\big(x(T)\big)$$

With respect to the appropriate variations $\delta x(t); \delta u(t); \delta\lambda(t);$

Result: *New* terminal co-state constraint:

$$\boxed{\lambda(T) = \frac{\partial V}{\partial x}(x(T)) + \frac{\partial \varphi}{\partial x}(x(T))\vec{\nu}}$$

Handling Additional Constraints

Example: what if *final time is not specified?*

- Must include the additional variation δT in the extremization process
- **Result:** $\boxed{H(T) = 0}$

Summary with (1) unspecified final time; (2) terminal constraints:

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T = f(x,u)$$ (PMP 1)

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T$$ (PMP 2)

$$\frac{\partial H}{\partial u} = 0\,; \quad or \quad H(x^*, u^*, \lambda^*) \leq H(x^*, u, \lambda^*)$$ (PMP 3)

$$\lambda(T) = \frac{\partial V}{\partial x}\big(x(T)\big) + \frac{\partial \varphi}{\partial x}\big(x(T)\big)\vec{v}$$ (PMP 4)

$$H(T) = 0$$ (PMP 5)

$$x(0) = x_0$$ (BC)

$$\varphi(x(T)) = 0$$ (TC)

Example: Bead on a Wire

Problem: bead moves without friction along a wire pushed by force u

$$m\,\ddot{x} = u$$

- **Goal:** move bead from x_0 to x_F in minimum time
- **Constraints:** $x(0) = x_0\,;\quad x(\mathrm{T}) = x_\mathrm{F}\,;\quad \dot{x}(0) = \dot{x}(T) = 0;\ \ |u| \le \mathrm{C}$
- **Terminal Constraints:**

Solution: PMP with unspecified final time & terminal constraints

- **Step 1:** convert dynamics to first order form: $z = (z_1\ z_2)^\mathrm{T}$

$$\dot{z} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u \equiv Az + Bu$$

- Terminal Constraints: $\psi_1(\mathrm{z(T)}) = z_1(\mathrm{T})\ -\ x_\mathrm{F} = 0$
- **Step 2:** construct the Hamiltonian for $J(x,u) = \int_0^T 1\, dt$

$$H = 1 + \lambda^T f(x,u) = 1 + \begin{pmatrix} \lambda_1 & \lambda_2 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u \right] = 1 + \lambda_1 z_2 + \left(\frac{\lambda_2}{m}\right) u$$

Bead on a Wire (continued)

- **Step 3:** Apply (PMP 2), the adjoint equations:

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \Rightarrow -\begin{pmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\end{pmatrix} = \begin{pmatrix}\partial H/\partial z_1\\ \partial H/\partial z_2\end{pmatrix} = \begin{pmatrix}0\\ \lambda_1\end{pmatrix} \Rightarrow \boxed{\begin{array}{l}\lambda_1(t) = \alpha\\ \lambda_2(t) = -\alpha t + \beta\end{array}}$$

- **Step 4:** Apply (PMP 3), the minimum principle:
  - For constrained control, must minimize H w.r.t. control u

$$H = 1 + \lambda_1 z_2 + \left(\frac{\lambda_2(t)}{m}\right) u$$

$$\Downarrow$$

$$u = \begin{cases} +C & sgn(\lambda_2(t)) < 0\\ -C & sgn(\lambda_2(t)) > 0\\ 0 & \lambda_2(t) = 0\end{cases} \Rightarrow \boxed{u = -C\, sgn(\lambda_2(t))}$$

  - Hence u(t) is a “switching control” with λ_2(t) the “switching function
  - The switching function is ***linear***, implying ***one*** switch.

Bead on a Wire (continued)

– **Step 5:** Apply (PMP 4), the adjoint terminal constraints:

$$\lambda(T) = \frac{\partial V}{\partial z}\big(z(T)\big) + \frac{\partial \varphi}{\partial z}\big(z(T)\big)\vec{v}$$

$$\Downarrow$$

$$\begin{pmatrix} \lambda_1(T) \\ \lambda_2(T) \end{pmatrix} = \begin{pmatrix} \partial\varphi_1/\partial z_1 & \partial\varphi_2/\partial z_1 \\ \partial\varphi_1/\partial z_2 & \partial\varphi_2/\partial z_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ -\alpha T + \beta \end{pmatrix}$$

$$\Downarrow$$

$$\beta = v_2 + v_1 T \;\Rightarrow\; \boxed{\lambda_1(t) = v_1 \qquad \lambda_2(t) = v_1(T - t) + v_2}$$

– **Step 6:** Apply (PMP 5), the undetermined final time condition:

$$H(T) = 1 + \lambda_1(T) z_2(T) + \left(\frac{\lambda_2(T)}{m}\right) u(T) = 1 + v_1 \cancelto{0}{\dot{x}(T)} - \left(\frac{C}{m}\right) \cancelto{v_2}{\lambda_2(T)}\, sgn\big(\cancelto{v_2}{\lambda_2(T)}\big)$$

$$= 1 - v_2 \left(\frac{C}{m}\right) sgn(v_2) = 0 \qquad \Rightarrow \qquad \boxed{v_2 = \pm\left(\frac{m}{C}\right)}$$

Bead on a Wire (continued)

- **Step 7:** Apply (PMP 1), the dynamics, and (BC)
  - We know that the control "switches" at some as yet unknown time, t_s
  - Integrate the acceleration to get the velocity over $t \in [0,T]$

$$\dot{x}(T) = \cancelto{0}{\dot{x}(0)} + \int_0^{t_s} \ddot{x}\,dt + \int_{t_s}^{T} \ddot{x}\,dt = \int_0^{t_s} -\frac{C}{m} sgn(\lambda_2(0))dt + \int_{t_s}^{T} \frac{C}{m} sgn(\lambda_2(0))dt$$

$$= \frac{C}{m} sgn(\lambda_2(0))[-t_s + (T - t_s)] = 0 \qquad \Rightarrow \qquad \boxed{t_s = \frac{T}{2}}$$

- **Step 8:** Apply (PMP 1), the dynamics, and (BC)
  - Integrate velocity to get position over $t \in [0,T]$, knowing switch at T/2
  - For constant acceleration: $x(t) = x(0) + \dot{x}(0)t + \ddot{x}(0)t^2/2$

$$x(T/2) = x(0) + \frac{\cancelto{0}{\dot{x}(0)T}}{2} - \left(\frac{C}{m}\right) sgn(\lambda_2(0))T^2/8$$

$$x(T) = x\left(\frac{T}{2}\right) + \dot{x}\left(\frac{T}{2}\right)\frac{T}{2} + \left(\frac{C}{m}\right) sgn(\lambda_2(0))\frac{T^2}{8} = x_0 - \left(\frac{C}{m}\right) sgn(\lambda_2(0))\frac{T^2}{4} = x_F$$

Bead on a Wire (continued)

– **Step 8:** (*continued*)

$$\Rightarrow \quad T = 2\sqrt{\frac{x_F - x_0}{-(\frac{C}{m})sgn(\lambda_2(0))}} = \boxed{2\sqrt{\frac{m|x_F - x_0|}{C}}} \qquad \boxed{sgn(\lambda_2(0)) = -sgn(x_F - x_0)}$$

– **Step 9:** Using the switching function characteristic to find v_1

$$\lambda_2(T/2) = 0 = v_1\left(\frac{T}{2}\right) + v_2 \quad \Rightarrow \quad v_1 = -\left(\frac{2}{T}\right)v_2 = \mp\left(\frac{2}{T}\right)\left(\frac{m}{C}\right)$$

$$\lambda_2(t) = v_1(T - t) + v_2 = \mp\left(\frac{m}{C}\right)\left[\left(\frac{2}{T}\right)(T - t) - 1\right] = \pm\left(\frac{m}{C}\right)\left[\frac{2t}{T} - 1\right]$$

$$\Rightarrow \quad \boxed{\mathrm{u}(t) = -\mathrm{C}\, sgn\left[sgn(x_F - x_0)\left(\frac{m}{C}\right)\left[\frac{2t}{T} - 1\right]\right]}$$

$$\Rightarrow \quad \mathrm{u}(t) = -\mathrm{C}\, sgn\left[(x_F - x_0)\left[\frac{2t}{T} - 1\right]\right]$$

Simpler, but equivalent

Example: Bang-Bang Control

Consider time optimal control of linear system

- $\dot{x} = Ax + Bu$
- $x(0) = x_0$; $x(T) = x_F$; $\varphi(x(T)) = x_F - x(T)$
- $|u| \leq 1$; $J = \int_0^T 1\, dt$ ← "minimum time control"

Apply PMP:

- $H = L + \lambda^T f = 1 + \lambda^T (Ax + Bu) = 1 + (\lambda^T A)x + (\lambda^T B)u$
- $\dot{x} = \left(\frac{\partial H}{\partial x}\right)^T = Ax + Bu$
- $u = \mathrm{argmin}(H) = -\,\mathrm{sgn}(\lambda^T B)$
- I.e., control u is +1 or -1 in value

Since H is linear w.r.t. u,
minimization occurs at boundary