Analysis of Basic PMP Solution

Find control u(t) which minimizes the function:

$$J(x,u) = \int_0^T L(x,u)dt \; + \; V\big(x(T)\big)$$

$x(t{=}0) = x_{0}$,
T fixed,
$\dot{x} = f(x,u)$

To find a necessary condition to minimize $J(x,u)$, we *augment* the cost function with the dynamics constraints

$$\tilde{J}(x,u,\lambda) = J(x,u) + \int_0^T \lambda^T(f(x,u) - \dot{x})dt \; = \int_0^T (H(x,u) - \lambda^T\dot{x})\, dt \; + V(x(T))$$

Extremize the augmented cost w.r.t. variations $\delta x(t)$; $\delta u(t)$; $\delta\lambda\,(t)$;

$$\delta\tilde{J} = \int_0^T \left[\overbrace{\left(\frac{\partial H}{\partial x} + \lambda^T\dot{x}\right)}^{0}\delta x + \overbrace{\frac{\partial H}{\partial u}}^{0}\delta u + \overbrace{\left(\frac{\partial H}{\partial \lambda} - \dot{x}^T\right)}^{0}\delta\lambda \right] \mathrm{dt}$$

$$+ \underbrace{\left(\frac{\partial V}{\partial x} - \lambda^T(T)\right)}_{0}\delta x(T) + \lambda^T(0)\,\underbrace{\delta x(0)}_{0}$$

$$H(x,u) \equiv L(x,u) + \lambda^T f(x,u)$$

Dimension of PMP Solution

Unknowns

- n state variables x(t)
- n co-state variables λ(t)
- m control variables u(t)

Optimal control satisfies:

$$\left.\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T = f(x,u) \qquad x(0) = x_0 \right\} \; n \text{ o.d.e.s with I.C.s}$$

$$\left.-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \frac{\partial V}{\partial x}\big(x(T)\big) \right\} \; n \text{ o.d.e.s with T.C.s}$$

2 point B.V. problem

$$\left.\frac{\partial H}{\partial u} = 0\,; \right\} \; m \text{ algebraic equations}$$

Additional Constraints?

Example: what if *final state* constraints $\phi_i(x(T))=0$, $i=1,\ldots,p$ are desired:

– Add constraints to cost using additional Lagrange multipliers

Multipliers

$$J_a(x,u) = \int_0^T L(x,u)dt + V\big(x(T)\big) + \underbrace{v_1\varphi_1\big(x(T)\big) + \ldots + v_p\varphi_p\big(x(T)\big)}_{\text{Cost augmentation}}$$

Extremizing the constrained, augmented the cost function yields

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T = f(x,u) \qquad x(0) = x_0$$

n o.d.e.s with I.C.s

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T \qquad \lambda(T) = \frac{\partial V}{\partial x}(x(T)) + \frac{\partial \varphi}{\partial x}(x(T))\vec{v}$$

n o.d.e.s with T.C.s

$$\frac{\partial H}{\partial u} = 0\ ;$$

m algebraic equations

$$\varphi_1\big(x(T)\big) = 0;\ \ldots;\ \varphi_p\big(x(T)\big) = 0$$

p algebraic equations

Additional Constraints?

Example: what if *final time* is undetermined?

- Final time T is an additional variable
- H(T) = 0 gives ***one*** additional constraint equation!