Minimum Variance Design

Estimator is a random function

- Takes measurements $y_1, y_2, \cdots, y_n$ as input, and produces a random variable, $\hat{X}_n$, with $\hat{x}_n$ as a specific estimate.
- The variance associated with the estimator is the “uncertainty” in the estimate. Minimum variance design is the “least uncertain”

Minimum Variance Design (Kalman 1960)

- The minimum variance estimate of state X given measurements Y is given by the *conditional mean* of X given Y.

$$\hat{x} = E[X|Y = y] = \int_{-\infty}^{\infty} x \, p(x|y) dx$$

With jointly Gaussian state (x) and measurements (y),

- *Mean:* $E\begin{bmatrix}\vec{x}\\ \vec{y}\end{bmatrix} = E\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix}$ Variance: $\begin{bmatrix}\Sigma_{xx} & \Sigma_{xy}\\ \Sigma_{yx} & \Sigma_{yy}\end{bmatrix}$, $\Sigma_{xy} = \Sigma_{yx}^T$
- The minimum variance estimate of $\vec{x}$ given $\vec{y}$

$$\hat{x} = \bar{x} + \Sigma_{xy}\Sigma_{yy}^{-1}(\vec{y} - \bar{y}) \qquad \Sigma_{x|y} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}^T$$

Recursive Construction of Kalman Filter (KF)

Initial system state, x_0, is Gaussian distributed: $x_0 \sim N(\bar{x}_0, P_{00})$

- Assume measurement $y_0 = H_0 x_0 + \omega_0$. Then x_0, y_0 are joint Gaussian:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \sim N\left(\begin{bmatrix} \bar{x}_0 \\ H_o \bar{x}_0 \end{bmatrix}, \begin{bmatrix} P_{00} & P_{00} H_0^T \\ H_0 P_{00} & H_0 P_{00} H_0^T + R_0 \end{bmatrix} \right)$$

- The *estimate* of state x_0 at time t_0 given measurements at t_0 is
  - $\hat{x}_{0|0} = \bar{x}_0 + P_{00} H_0^T (H_0 P_{00} H_0^T + R_0)^{-1} (y_0 - H_o \bar{x}_0)$
- Because the estimate is a conditional mean, the associated variance is:
  - $\Sigma_{0|0} = P_{00} - P_{00} \mathrm{H}_0^{\mathrm{T}} \left(\mathrm{H}_0 \mathrm{P}_{00} \mathrm{H}_0^{\mathrm{T}} + \mathrm{R}_0 \right)^{-1} \mathrm{H}_0 \mathrm{P}_{00}$

Let's *propagate* the state estimate to time t_1

- The system equation is: $x_1 = A_0 x_0 + B_0 u_0 + G_0\, \eta_0$
- If a Gaussian CRV is substituted into the x_0"slot," then $x_1$will be a Gaussian CRV with
  - $\hat{x}_{1|0} = \mathrm{E}\left[\mathrm{A}_0 \hat{\mathrm{x}}_{0|0} + \mathrm{B}_0 \mathrm{u}_0 \mathrm{G}_0 \eta_0 \right] = A_0 \hat{\mathrm{x}}_{0|0} + B_0 u_0$
  - $\Sigma_{1|0} = A_0 \Sigma_{0|0} A_0^T + G_0 Q_0 G_0^T$

Recursive Construction of KF

The *predicted* measurement at time t_1, and its uncertainty:

- $\hat{x}_{k+1|k} = A_k \hat{x}_{k+1|k} + B_k u_k$
- $cov(\hat{\mathrm{y}}_{1|0}, \hat{\mathrm{y}}_{1|0}) = H_1 \Sigma_{1|0} H_1^T + R_1$
- $\begin{bmatrix} \hat{x}_{1|0} \\ \hat{y}_{1|0} \end{bmatrix} \sim N\left(\begin{bmatrix} \hat{x}_{1|0} \\ H_1 \hat{x}_{1|0} \end{bmatrix}, \begin{bmatrix} \Sigma_{1|0} & \Sigma_{1|0} H_1^T \\ H_1 \Sigma_{1|0} & H_1 \Sigma_{1|0} H_1^T + R_1 \end{bmatrix} \right)$

Now in corporate a measurement time t_1, and use formulae for conditional mean and its variance

- $\hat{x}_{1|1} = \hat{x}_{1|0} + \Sigma_{1|0} H_1^T \left(H_1 \Sigma_{1|0} H_1^T + R_1 \right)^{-1} (y_1 - H_1 \hat{x}_{1|0})$
- $\Sigma_{1|1} = \Sigma_{1|0} - \Sigma_{1|0} \mathrm{H}_1^\mathrm{T} \left(\mathrm{H}_1 \Sigma_{1|0} \mathrm{H}_1^\mathrm{T} + \mathrm{R}_1 \right)^{-1} \mathrm{H}_1 \Sigma_{1|0}$

Recursive Construction of KF

By induction, the KF has a ***2-step*** structure:

- Dynamic (time) update)
  - $\hat{x}_{k+1|k} = A_k \hat{x}_{k+1|k} + B_k u_k$
  - $\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q_k G_k^T$
- Measurement Update
  - $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \underbrace{K_{k+1}}_{\text{“Kalman Gain”}} \overbrace{\left(y_{k+1} - H_{k+1}\hat{x}_{k+1|k}\right)}^{\text{“residual,” “innovation”}}$
  - $\Sigma_{k+1|k+1} = \Sigma_{k+1|k} - \Sigma_{k+1|k} \mathrm{H}_{k+1}^{\mathrm{T}} \left(\mathrm{H}_{k+1} \Sigma_{k+1|k} \mathrm{H}_{k+1}^{\mathrm{T}} + \mathrm{R}_{k+1}\right)^{-1} \mathrm{H}_{k+1} \Sigma_{k+1|k}$

$$= (I - K_{k+1} H_{k+1}) \Sigma_{k+1|k} = \Sigma_{k+1|k} (\mathrm{I} - \mathrm{H}_{k+1}^{\mathrm{T}} \mathrm{K}_{k+1}^{\mathrm{T}})$$

Where the “Kalman Gain” is:

- $K_{k+1} = \Sigma_{k+1} H_k^T \left(\underbrace{\mathrm{H}_{k+1} \Sigma_{k+1|k} \mathrm{H}_{k+1}^{\mathrm{T}}}_{\text{“How much do I trust the model?”}} + \underbrace{\mathrm{R}_{k+1}\Big)^{-1}}_{\text{“How much do I trust the measurements?”}}\right.$