Fixed Lag Smoothing

(see Anderson & More 7.3)

Goal: is to develop an estimator for $\hat{x}_{k-N|k}$ for some N, assuming

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$$x_{k+1} = A_k x_k + B_k u_k + \eta_k$$
$$y_k = H_k x_k + \omega_k$$

(with zero mean white Gaussian noise). N is the ***time lag.*** I.e., we want to develop an estimate of the state N time steps ago, using all measurements up until current time t_k

Why:

- We will show that $\Sigma_{k-N|k} \leq \Sigma_{k-N|k-N}$
- Use measurements $t_{k-N+1}, t_{k-N+2}, \ldots, t_k$ to improve estimation accuracy, at the expense of a *time lag* in the estimate
- The method can also produce *smoothed* estimates for x at $t_{k-N+1}, t_{k-N+2}, \ldots, t_k$, i.e., at states within the "smoothing window"
- The methodology can be useful template for other problems.

Background: the 1-step KF

Recall the 2-step KF:

- Dynamic (time) update
  - $\hat{x}_{k+1|k} = A_k \hat{x}_{k+1|k} + B_k u_k$
  - $\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q_k G_k^T$
- Measurement Update
  - $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(y_{k+1} - H_{k+1}\hat{x}_{k+1|k}\right)$
  - $\Sigma_{k+1|k+1} = \Sigma_{k+1|k}\left(\mathrm{I} - \mathrm{H}_{k+1}^{\mathrm{T}}\mathrm{K}_{k+1}^{\mathrm{T}}\right)$ $\quad K_{k+1} = \Sigma_{k+1|k} H_k^T \left(\mathrm{H}_{k+1}\Sigma_{k+1|k}\mathrm{H}_{k+1}^{\mathrm{T}} + \mathrm{R}_{k+1}\right)^{-1}$

The 1-step KF:

- $\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k = A_k\left[\hat{x}_{k|k-1} + K_k\left(y_k - H_k \hat{x}_{k|k-1}\right)\right]$

  $= (A_k - K_k H_k)\,\hat{x}_{k|k-1} + L_k y_k$ $\quad$ *where* $L_k = A_k K_k$

- $\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q_k G_k^T = A_k \Sigma_{k|k-1}\left(\mathrm{I} - \mathrm{H}_{\mathrm{k}}^{\mathrm{T}}\mathrm{K}_{\mathrm{k}}^{\mathrm{T}}\right) A_k^T + G_k Q_k G_k^T$

  $= A_k \Sigma_{k|k-1}(\mathrm{A}_{\mathrm{k}} - L_k H_k)^T + G_k Q_k G_k^T$

The Augmented State Vector

Define an ***Augmented* dynamical system:**

$$\vec{X}^a_{k+1} = \begin{bmatrix} x^{(0)}_{k+1} \\ x^{(1)}_{k+1} \\ x^{(2)}_{k+1} \\ \vdots \\ x^{(N+1)}_{k+1} \end{bmatrix} = \begin{bmatrix} A_k & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} x^{(0)}_{k} \\ x^{(1)}_{k} \\ x^{(2)}_{k} \\ \vdots \\ x^{(N+1)}_{k} \end{bmatrix} + \begin{bmatrix} G_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_k$$

$$y_k = [H_k \;\; 0 \;\; 0 \;\; \cdots \;\; 0]\vec{X}^a_k + \omega_k$$

Where matrices A_k, G_k, H_k and noises η_k, ω_k come from original system.

The augmented state is a vector of ***time-shifted*** states

$x^{(0)}_{k+1} = A_k x^{(0)}_k + G_k \eta_k = A_k x_k + G_k \eta_k =$ system state at t_{k+1}

$x^{(1)}_{k+1} = x^{(0)}_k = x_k =$ system state at t_k

$x^{(2)}_{k+1} = x^{(1)}_k = x^{(0)}_k = x_{k-1} =$ system state at t_{k-1}

The Augmented State Vector

$$\vdots$$

$$x_{k+1}^{(i)} = x_k^{(i-1)} = \cdots = x_{k-i+1} = \text{system state at } t_{k-i+1}$$

$$\vdots$$

$$x_{k+1}^{(N+1)} = x_k^{(N)} = x_{k-1}^{(N-1)} = \cdots = x_{k-N} = \text{system state at } t_{k-N}$$

So, the last component of the augmented state (and its associated covariance) is the state for which we wish to produce a smoothed estimate (and its associated estimate uncertainty).

Key Idea: Since the augmented system satisfies all of the requirements for a discrete time linear system with zero-mean, white Gaussian noise, we can construct a Kalman filter for the augmented system, and then isolate the terms of interest in the resulting filter equations.

The Variance Reduction Property

$$\Sigma_{k-N|k} = \Sigma_{k-N|k-N-1} - \sum_{l=k-N}^{k} \Sigma_{l|l-1}^{(q,0)} H_l^T \left[H_l \Sigma_{l|l-1} H_l^T + R_l\right]^{-1} H_l \Sigma_{l|l-1}^{(q,0)^T}$$

where: $q = l - k + N$

Reduction in estimate uncertainty due to "smoothing"