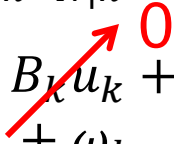


# Fixed Lag Smoothing

(see Anderson & More 7.3)

**Goal:** is to develop an estimator for  $\hat{x}_{k-N|k}$  for some N, assuming

$$\begin{aligned}x_{k+1} &= A_k x_k + B_k u_k + \eta_k \\ y_k &= H_k x_k + \omega_k\end{aligned}$$


(with zero mean white Gaussian noise). N is the **time lag**. I.e., we want to develop an estimate of the state N time steps ago, using all measurements up until current time  $t_k$

**Why:**

- We will show that  $\Sigma_{k-N|k} \leq \Sigma_{k-N|k-N}$
- Use measurements  $t_{k-N+1}, t_{k-N+2}, \dots, t_k$  to improve estimation accuracy, at the expense of a *time lag* in the estimate
- The method can also produce *smoothed* estimates for  $x$  at  $t_{k-N+1}, t_{k-N+2}, \dots, t_k$ , i.e., at states within the “smoothing window”
- The methodology can be useful template for other problems.

# Background: the 1-step KF

## Recall the 2-step KF:

– Dynamic (time) update)

- $\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k$
- $\Sigma_{k+1|k} = A_k \Sigma_{k|k} A_k^T + G_k Q_k G_k^T$

– Measurement Update

- $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} (y_{k+1} - H_{k+1} \hat{x}_{k+1|k})$
- $\Sigma_{k+1|k+1} = \Sigma_{k+1|k} (I - H_{k+1}^T K_{k+1}^T) \quad K_{k+1} = \Sigma_{k+1|k} H_{k+1}^T (H_{k+1} \Sigma_{k+1|k} H_{k+1}^T + R_{k+1})^{-1}$

## The 1-step KF:

$$\begin{aligned} - \hat{x}_{k+1|k} &= A_k \hat{x}_{k|k} + B_k u_k = A_k [\hat{x}_{k|k-1} + K_k (y_k - H_k \hat{x}_{k|k-1})] \\ &= (A_k - K_k H_k) \hat{x}_{k|k-1} + L_k y_k \quad \text{where } L_k = A_k K_k \end{aligned}$$

$$\begin{aligned} - \Sigma_{k+1|k} &= A_k \Sigma_{k|k} A_k^T + G_k Q_k G_k^T = A_k \Sigma_{k|k-1} (I - H_k^T K_k^T) A_k^T + G_k Q_k G_k^T \\ &= A_k \Sigma_{k|k-1} (A_k - L_k H_k)^T + G_k Q_k G_k^T \end{aligned}$$

# The Augmented State Vector

Define an **Augmented** dynamical system:

$$\vec{X}_{k+1}^a = \begin{bmatrix} x_{k+1}^{(0)} \\ x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \\ \vdots \\ x_{k+1}^{(N+1)} \end{bmatrix} = \begin{bmatrix} A_k & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} x_k^{(0)} \\ x_k^{(1)} \\ x_k^{(2)} \\ \vdots \\ x_k^{(N+1)} \end{bmatrix} + \begin{bmatrix} G_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta_k$$

$$y_k = [H_k \ 0 \ 0 \ \cdots \ 0] \vec{X}_k^a + \omega_k$$

Where matrices  $A_k, G_k, H_k$  and noises  $\eta_k, \omega_k$  come from original system.

The augmented state is a vector of **time-shifted** states

$$x_{k+1}^{(0)} = A_k x_k^{(0)} + G_k \eta_k = A_k x_k + G_k \eta_k = \text{system state at } t_{k+1}$$

$$x_{k+1}^{(1)} = x_k^{(0)} = x_k = \text{system state at } t_k$$

$$x_{k+1}^{(2)} = x_k^{(1)} = x_k^{(0)} = x_{k-1} = \text{system state at } t_{k-1}$$

# The Augmented State Vector

⋮

$$x_{k+1}^{(i)} = x_k^{(i-1)} = \cdots = x_{k-i+1} = \text{system state at } t_{k-i+1}$$

⋮

$$x_{k+1}^{(N+1)} = x_k^{(N)} = x_{k-1}^{(N-1)} = \cdots = x_{k-N} = \text{system state at } t_{k-N}$$

So, the last component of the augmented state (and its associated covariance) is the state for which we wish to produce a smoothed estimate (and its associated estimate uncertainty).

**Key Idea:** Since the augmented system satisfies all of the requirements for a discrete time linear system with zero-mean, white Gaussian noise, we can construct a Kalman filter for the augmented system, and then isolate the terms of interest in the resulting filter equations.

# The Variance Reduction Property

$$\Sigma_{k-N|k} = \Sigma_{k-N|k-N-1} - \sum_{l=k-N}^k \underbrace{\Sigma_{l|l-1}^{(q,0)} H_l^T [H_l \Sigma_{l|l-1} H_l^T + R_l]^{-1} H_l \Sigma_{l|l-1}^{(q,0)T}}_{\text{Reduction in estimate uncertainty due to "smoothing"}}$$

where:  $q = l - k + N$

Reduction in estimate uncertainty  
due to "smoothing"