**Problem 1 Statement:** This problem explores the difference between a simple finite horizon optimal control problem and its infinite horizon optimal control counterpart. Consider a system whose dynamics are governed by the equations $\dot{x} = ax + bu$, where $x \in \mathbb{R}$ denotes the state, $u \in \mathbb{R}$ is a single scalar control input, and $a$ and $b$ are constant *positive* scalars: $a = 2$, and $b = 0.5$.

Consider the optimal control problem with cost: $J = (1/2)\int_{t_0}^{T} u^2(t)\ dt + (1/2)cx^2(T)$, where final time $T$ is given and $c > 0$ is a constant. The optimal control for finite time $T > 0$ is derived in Example 2.2 in OBC. Now consider the infinite horizon problem with cost $J = (1/2)\int_{t_0}^{\infty} u^2(t) + cx^2(t)\ dt$

**Part (a):** Solve the algebraic Ricatti equation to find $P$, leading to the optimal control$u^*(t) = -bPx^*(t)$ for the infinite horizon case.

**Solution:** With this choice of cost function, we define the Hamiltonian for the scalar system to be $H = (1/2)u^2 + (1/2)cx^2 + \lambda(ax + bu)$. Using the *algebraic Ricatti equation (ARE)* will give:

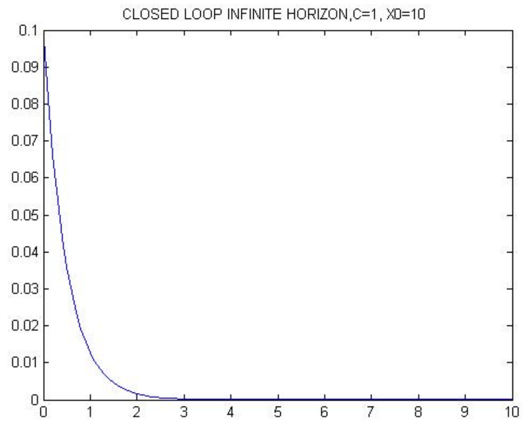
$$p^2 - p(\frac{2a}{b^2}) - \frac{c}{b^2} = 0 \tag{1}$$

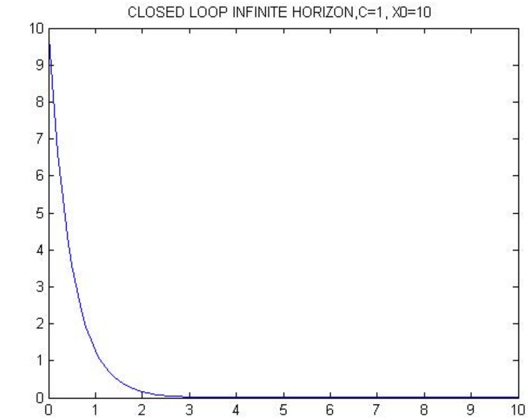
The solution to this equation is not unique, however the intent of the LQR is to stabilize the system so we must choose the value of $p$ that stabilizes the closed-loop system. Consider $\dot{x}(t) = ax(t) + b * (-bpx(t)) = (a - b^2p)x(t)$. A stable dynamical system is one that we can express as $\dot{x}(t) = rx(t)$ for some $r < 0$. This motivates us to choose:

$$p = \frac{a}{b^2} + \frac{1}{b}\sqrt{\frac{a^2}{b^2} + c} \tag{2}$$

This yields the dynamical system: $\dot{x}(t) = -(\sqrt{a^2 + b^2c})x(t)$

**Part (b):** For the two initial conditions $x(t_0) = 0.1$ and $x(t_0) = 10.0$, plot the closed loop system response for the infinite horizon system over an interval of 10 seconds.






**Part (c):** Plot the closed loop system response of the *finite horizon* optimal controller for the case of $c = 0.1$ and $c = 10.0$. Plot the response for finite time horizons $T = 1$ and $T = 10$. Also, plot the gains as a function of time.

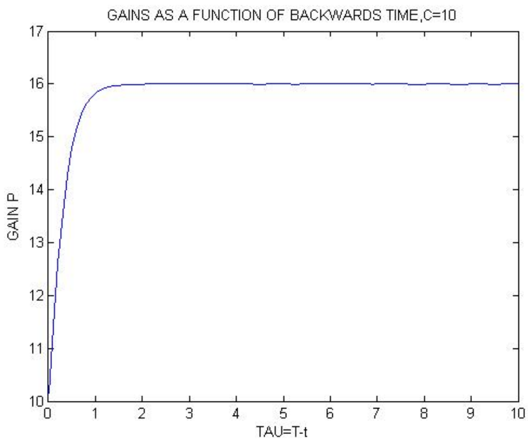
The closed loop system responses can be plotted by simulating the example in OBC. The gains as a function of time can be found via deriving the Ricatti equation for this scalar system, which gives rise to:

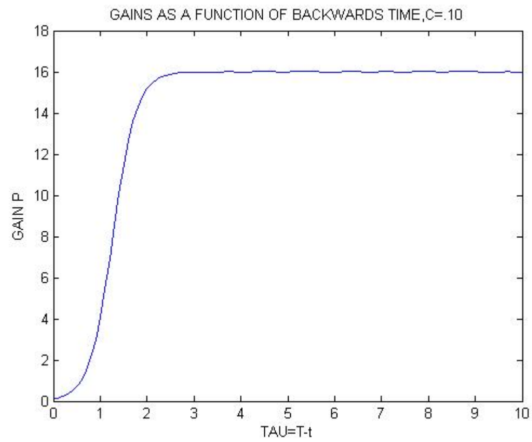
$$\begin{aligned} \frac{dp}{dt} &= p^2b^2 - 2pa \\ p(T) &= c \end{aligned} \tag{3}$$

A variable substitution $\tau = T - t$ can be made to convert this to a forward time differential equation, yielding:

$$\begin{aligned} \frac{dp}{d\tau} &= -p^2b^2 + 2pa \\ p(\tau = 0) &= c \end{aligned} \tag{4}$$

The plots for the case $T = 10$, $x_0 = 0.1$ are as follows:






**Part (d):** Compare the infinite horizon and finite horizon optimal control solutions. Which finite time solution is the closest to the infinite time solution? How do the gains differ?

**Problem 2 Statement:** Consider the following optimal control problem. You are given a linear dynamical system

$$\dot{x} = Ax + Bu \tag{5}$$

You wish to design the optimal control $u(t)$ which optimizes the following performance index:

$$J(x, u) = (1/2)\int_0^T \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} Q & V \\ V^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt + (1/2)x^T(T)P_T x(T) \tag{6}$$

If we define the modified Kalman gain as $K = R^{-1}(V^T + B^T P)$ then show that the optimal control for this problem is given by $u(t) = -K(t)x(t)$ where $P$ is the solution to the following Ricatti type equation: $-\dot{P} = PA + A^T P - K^T RK + Q$ with terminal condition $P(T) = P_T$.

**Solution:** The solution strategy to this derivation is to follow the basic framework of the Pontryagin Maximum Principle (PMP) specialized to this particular cost functional and dynamic system. Thus, the first step is to formulate the Hamiltonian:

$$H = (1/2)x(t)^T Qx(t) + (1/2)u(t)^T Ru(t) + x(t)^T Vu(t) + \lambda(t)^T(Ax(t) + Bu(t)) \tag{7}$$

The implied assumption is that $A, B, Q, R$ and $V$ are not time-dependent quantities, although in general they could be. We can obtain the optimal controller's form by recourse to the PMP condition that applies to system's without control constraints, i.e. $\frac{\partial H}{\partial u} = 0$. This condition yields:

$$Ru(t) + V^T x(t) + B^T\lambda(t) = 0 \Rightarrow u^*(t) = -R^{-1}(V^T x(t) + B^T\lambda(t)) \tag{8}$$

Similar to the derivation for the LQR controller, we postulate that the co-state variable has the separable form: $\lambda(t) = P(t)x(t)$. Substituting this form into (6) yields:

$$\boxed{u^*(t) = -R^{-1}(V^T + B^T P(t))x(t) \equiv -K(t)x(t)} \tag{9}$$

Now we must show that the matrix $P(t)$ satisfies the Ricatti-type equation given. We can do this by invoking the PMP condition governing the co-state evolution, i.e. $\frac{\partial H}{\partial x} = -\dot{\lambda}$. Evaluating leads to:

$$-\dot{\lambda} = Qx(t) + Vu(t) + A^T\lambda(t) \tag{10}$$

Using the assumed form of $\lambda(t)$ and the derived form of $u^*$, and the dynamical equation $\dot{x} = Ax(t) + Bu(t)$, we arrive at:

$$Qx(t) - VK(t)x(t) + A^TP(t)x(t) = -\dot{P}(t)x(t) - P(t)Ax(t) + P(t)BK(t)x(t) \tag{11}$$

Since this equation must hold for all possible state values, we can rearrange the expression and group terms to obtain the following equation (explicit time-dependency has been suppressed with the understanding that matrices P and K are functions of time)

$$-\dot{P} = PA + A^TP + Q - (V + PB)K \tag{12}$$

Now, after some algebraic manipulation and the symmetrical form of $P$ and $R$, we can show that $(V + PB) = K^TR$, which allows us to write:

$$\boxed{-\dot{P} = PA + A^TP + Q - K^TRK} \tag{13}$$

The terminal condition is found by using the terminal co-state equation from PMP: $\lambda(T) = \frac{\partial V}{\partial x}$, where $V(x(T))$ is understood to be the terminal penalty term present. This yields $\lambda(T) = P(T)x(T) = P_Tx(T)$ and again due to the applicability of this equation for arbitrary final state, we can claim that $\boxed{P(T) = P_T}$, concluding the exercise.