CDS 112 Winter 2014/2015
Solution to Homework 3

Problem 1(a) Statement: Use the maximum principle to show that the shortest path between two points in the plane is a straight line. To do this, model a control system

$$\dot{x} \;=\; u$$

where $x \in \mathbb{R}^2$ is the position of points in the plane and $u \in \mathbb{R}^2$ is the velocity of a point along a curve. Find the path of minimal length connecting $x(0) = x_0$ and $x(1) = x_f$. To minimize the length of the curve, let the cost along the path be

$$J \;=\; \int_0^1 ||\dot{x}||\; dt \;=\; \int_0^1 \sqrt{\dot{x}^T \dot{x}}\; dt$$

subject to the initial and final position constraints.

Solution. Since the dynamics can be written in the form $\dot{x}_1 = f_1 = u_1$ and $\dot{x}_2 = f_2 = u_2$, the Hamiltonian can be expressed as:

$$\begin{aligned} H(x,u) &= L(x,u) \;+\; \lambda^T f \;=\; ||\dot{x}|| + \;\lambda_1 f_1 \;+\; \lambda_2 f_2 \\ &= \sqrt{u_1^2 \;+\; u_2^2} \;+\; \lambda_1 u_1 + \;\lambda_2 u_2. \end{aligned}$$

Now apply the Pontryagin conditions. The adjoint state equations are:

$$-\dot{\lambda}_1 = \frac{\partial H}{\partial x_1} = 0 \qquad -\dot{\lambda}_2 = \frac{\partial H}{\partial x_2} = 0$$

from which we can conclude that both λ_1 and λ_2 are constants. Since the controls are *not* constrained, we can apply the simple principle that:

$$\frac{\partial H}{\partial u_1} = \frac{u_1}{\sqrt{u_1^2 + u_2^2}} + \lambda_1 = 0 \qquad \frac{\partial H}{\partial u_2} = \frac{u_1}{\sqrt{u_1^2 + u_2^2}} + \lambda_2 = 0.$$

These results imply that:

$$\frac{\lambda_1^*}{\lambda_2^*} = \frac{u_1^*}{u_2^*} = \frac{\dot{x}_1}{\dot{x}_2} = \frac{dx_1}{dx_2} = constant.$$

This last result implies that the slope of the trajectory connecting the two points is a constant, which implies that the trajectory connecting the start and final points is a straight line.

Problem 1(b) Statement: Use the Calculus of Variations to find the trajectory $y(x)$ which minimizes the cost $J = \int_0^1 \sqrt{1 + \dot{y}^2}\; dx$ subject to the conditions that $y(x = 0) = 0$ and $y(x = 1) = 1$.

Solution: (M. Burkhardt Feb. 2015) We will solve this problem in some generality to illustrate the procedure for a broad set of variational problems. It should be noted that Jost's *Calculus of Variations* is an excellent source on these methods. Essentially, we are trying to minimize a functional:

$$J(u(x)) = \int_a^b F(x, u(x), u'(x))\ dx \tag{1}$$

Subject to the boundary constraints $u(a) = u_1$ and $u(b) = u_2$. The idea is to introduce an arbitrary perturbation on the input function $u(x) \to u(x) + \epsilon\eta(x)$ and then minimize the functional with respect to the arbitrary perturbation ϵ. Upon making some reasonable smoothness assumptions, we can express this as:

$$\frac{d}{dx} J(u(x) + \epsilon\eta(x))|_{\eta=0} = \int_a^b \Big(\frac{\partial F}{\partial u} \cdot \eta(x) + \frac{\partial F}{\partial u'} \cdot \eta'(x)\Big)\ dx \tag{2}$$

At this point, you can use integration by parts, the boundary conditions, and the *fundamental lemma of the calculus of variations* (see Jost text) to conclude the *Euler-Lagrange* equations: For a function $u(x)$ that minimizes the above functional subject to the stated boundary conditions, then the function $u(x)$ must also satisfy the following system of second-order differential equations:

$$\frac{d}{dx}\Big(\frac{\partial F}{\partial u'}\Big) - \frac{\partial F}{\partial u} = 0 \tag{3}$$

Specialized to our particular problem, we have $F(y') = \sqrt{1 + (y')^2}$. Thus, we can invoke the Euler-Lagrange equations to claim that *any* solution that minimizes the functional must also satisfy:

$$\frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) = 0 \tag{4}$$

$$\begin{aligned} \frac{d}{dx}\Big(\frac{y'}{\sqrt{1+(y')^2}}\Big) &= 0 \\ \Rightarrow \frac{u''}{\sqrt{1+(u')^2}} - \frac{(u')^2 u''}{(\sqrt{1+(u')^2})^3} &= 0 \end{aligned} \tag{5}$$

Clearing the denominator yields:

$$\boxed{\frac{u''}{(\sqrt{1+(u')^2})^3} = 0 \Rightarrow u''(x) = 0} \tag{6}$$

This equation must hold for all solutions that minimize the functional we care about, and the expression must also hold for $x \in [a, b]$. Which kinds of functions have this property? *The solution has to be a straight line between the endpoints.*

Problem 3 Statement: In this problem you will use the Hamilton-Jacobi-Bellman equation to design a controller for the nonlinear system:

$$\ddot{x} = -x^3 + u \tag{7}$$

Here, $x \in \mathbb{R}$ is the system state and u is the control. This is a simplified model of a mechanical oscillator with a *hardening spring*. Design the control to minimize the cost function $J = \frac{1}{2}\int_0^\infty (x^2 + u^2).\ dt$.

Solution: (M. Burkhardt Feb. 2015)

The general Hamilton Jacobi Bellman (HJB) equation can be written:

$$-V_t(t,x) = \inf_u \{L(t,x,u(t)) + \langle V_x(t,x), f(t,x,u)\rangle\} \tag{8}$$

Where $\ddot{x} = f(x,u,t)$ and $\langle\rangle$ denotes the natural inner product. In our particular case, the ∞-horizon mandates that the optimal cost to go $V(x,t)$ not be an explicit function of t, i.e. $V = V(x) \Rightarrow \frac{\partial V}{\partial t} = 0$. Thus,

$$0 = \inf_u \{\frac{1}{2}(x^2 + u^2) + \frac{\partial V}{\partial x}^T (-x^3 + u)\} \tag{9}$$

Evaluating the minimum yields $u^* = -\frac{\partial V}{\partial x}$ and

$$0 = \frac{1}{2}x^2 - \frac{1}{2}\Big(\frac{\partial V}{\partial x}\Big)^2 - \frac{\partial V}{\partial x}x^3 \tag{10}$$

The instinct to solve this challenging nonlinear ODE is misleading– we already know that $u^* = -\frac{\partial V}{\partial x}$, and the above expression is an *algebraic* expression in $\frac{\partial V}{\partial x}$, so all we must do is simply solve for $\frac{\partial V}{\partial x}$. Similar to the standard LQR derivation for the scalar system, we must take care to select the positive choice of $\frac{\partial V}{\partial x}$ because then the closed-loop system will be stable (this is quite analagous to choosing the scalar p value that yields the stable closed-loop system in the scalar LQR examples). So, rerranging the above equation:

$$\begin{aligned} \Big(\frac{\partial V}{\partial x}\Big)^2 + (2x^3)\Big(\frac{\partial V}{\partial x}\Big) - x^2 = 0 \\ \Rightarrow \Big(\frac{\partial V}{\partial x}\Big) = -x^3 + x\sqrt{x^4+1} \end{aligned} \tag{11}$$

This gives rise to the optimal controller $\boxed{u^* = x(t)^3 - x(t)\sqrt{x(t)^4+1}}$, where the time-dependency has been included to stress that this procedure yields time-varying state feedback. It is clear that the closed loop system $\ddot{x}(t) = -x(t)\sqrt{x(t)^4+1}$ is in fact stable.