CDS 112: Winter 2014/2015

Solution #6

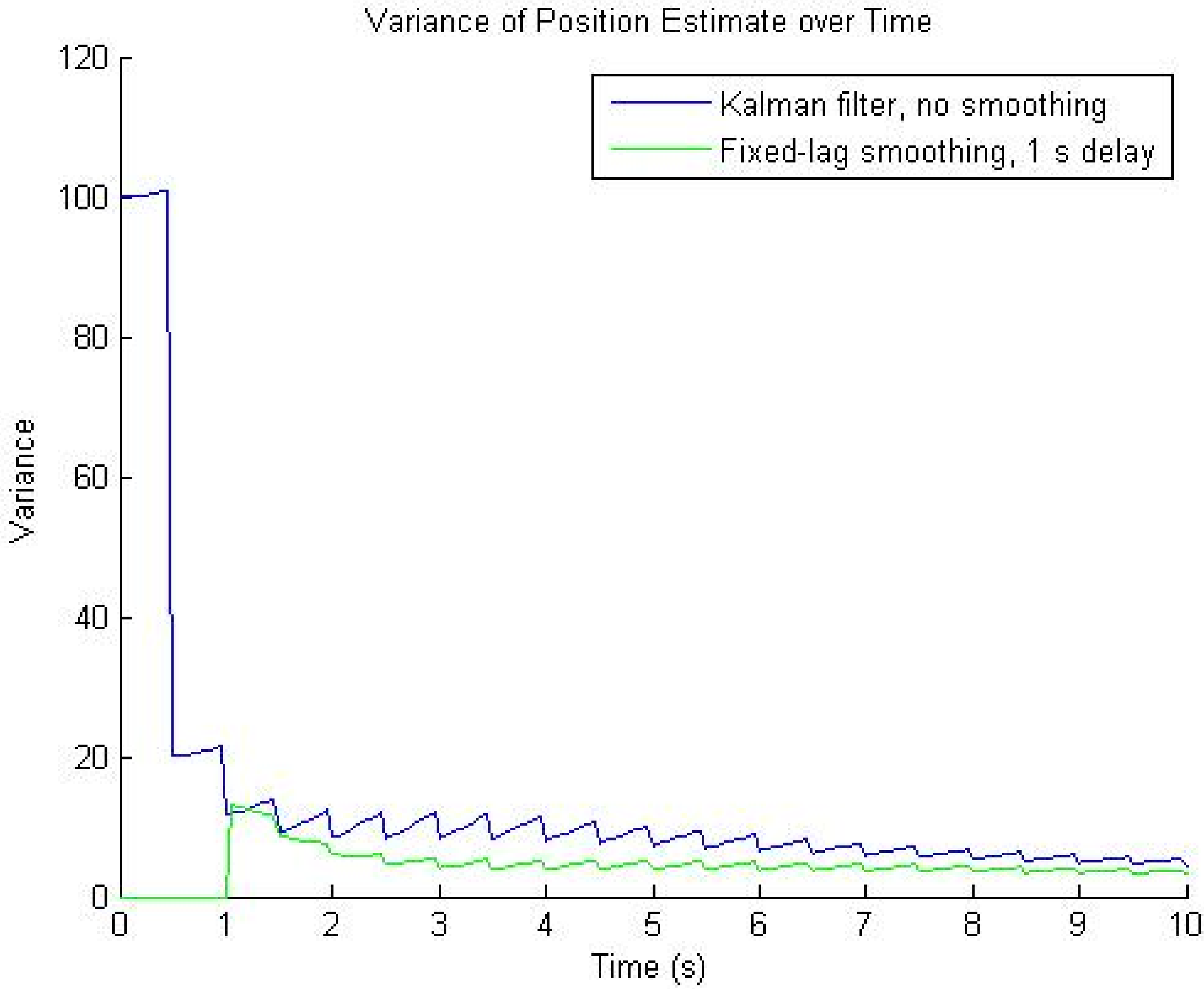
Problem #1: In this problem, you were to revisit the "Kidnapped Robot" problem from the previous week's homework. Fixed lag smoothing can improve the estimate of vehicle position at the cost of delay in making that information available. For the same system (using GPS and accelerometer information) described in Homework set #5, find the fixed lag smoothing equations for a smoother which delays the position estimate by four seconds. What is the covariance improvement of the fixed lag smoother over that of the Kalman filter?

Note: The original delay of two GPS measurement cycles of 4 seconds is inconsistent with the last assignment; two GPS cycles corresponds to a delay of one second.

The covariance update of the smoothed state is given by the following equation:

$$\Sigma^a_{k+1|k} = A^a_k \Sigma^a_{k|k-1}[A^a_k - L^a_k H^a_k]^T + G^a_k Q^a_k (G^a_k)^T \tag{1}$$

Here, the gain matrix L^a_k is given by: $L^a_k = A^a_k \Sigma^a_{k|k-1}(H^a_k)^T(H^a_k \Sigma^a_{k|k-1}(H^a_k)^T + R_k)^{-1}$. All of the required quantities in this expression were previously derived in the kidnapped robot problem. Keep in mind that the measurement matrix H_k changes when a GPS or IMU measurement is available. Likewise, the measurement covariance matrix R_k also changes with the timestep. Next, the covariance is initialized to $\Sigma_{0|1} = P_{00}$ and the covariance is updated and propogated using the equations listed in the Fixed Lag Smoothing notes. Implementing this procedure into MATLAB yields the following plot comparing the covariance of the fixed lag smoother versus the regular Kalman filter.



Problem #2:

Part (a): LQG In this problem you were asked to design a **steady-state** LQG controller, assuming that the cost is:

$$J = E\left\{ \int_0^\infty [x^T Q x + u^T R u] dt \right\}$$

where the state[1] x consists of:

$$x = \begin{bmatrix} (\theta - \theta_f) \\ \dot{\theta} \end{bmatrix}$$

[1]Note that the state can be written simply as $x = [\theta \; \dot{\theta}]^T$ if the θ-coordinates are adjusted so that $\theta_f = 0$.

where the weighting matrices Q and R take the form:

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = 1$$

and where a small disturbance due to a *gravity gradient* acts on the system in the following way

$$I\ddot{\theta} \;=\; u + \eta(t) \tag{2}$$

where $\eta(t)$ can be modeled by zero mean white Gaussian noise with covariance 0.001. Assume that only satellite orientation measurements are available for the estimator:

$$y(t) = \theta(t) + \omega(t) \tag{3}$$

and assume the measurement noise $\omega(t)$ is zero mean white Gaussiam with 0.5 degrees^2 variance.

Solution:

The dynamics can be converted to first-order form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ (1/I) \end{bmatrix} u \triangleq Ax + Bu \tag{4}$$

where $x = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.

LQR controller: Because the system in Equation (4) is linear, and the cost is quadratic, the **Steady-State** control law is given by the standard LQR steady-state feedback law applied to the state estimate

$$u(t) = -R^{-1}B^T P\hat{x}(t) = -1^{-1}\begin{bmatrix} 0 & (1/I) \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \hat{x}(t) = -\frac{1}{I}\begin{bmatrix} p_{12} & p_{12} \end{bmatrix} \hat{x} \tag{5}$$

where P is the solution to the algebraic Riccati Equation:

$$0 = PA + A^T P - PBR^{-1}B^T P + Q \tag{6}$$

and the state estimate, $\hat{x}$, is the solution to the Kalman-Bucy filter:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - H\hat{x})$$

with $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ (based on the measurement equation (3)), and the filter gain L to be analyzed below.

If you choose a particular value for the satellite inertia, I, then one could use the MATLAB `lqr`() function to design the feedback controller. Here we will carry out the synthesis directly, and as a function of I. First, let's substitute the relevant quantities into Equation (6):

$$\begin{aligned} 0 \;=\; & \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\ & - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 \\ (1/I) \end{bmatrix} 1^{-1} \begin{bmatrix} 0 & (1/I) \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

which reduces to the following algebraic equations for the components of P:

$$
\begin{aligned}
0 &= 10 - (1/I)^2 p_{12} && (7)\\
0 &= p_{11} - (1/I)^2 p_{12} p_{22} && (8)\\
0 &= 2p_{12} + 1 - (1/I)^2 p_{22}^2 && (9)
\end{aligned}
$$

From which we can derive:

$$p_{12} = \sqrt{10} I \qquad p_{22} = I\sqrt{2I\sqrt{10}+1}\ .$$

Hence, from (5) we see that the feedback control takes the form:

$$u = -K\hat{x} = -\begin{bmatrix} \sqrt{10} & \sqrt{2I\sqrt{10}+1} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}.$$

If, for example, we choose $I = 1$, then the gain matrix is (up to 4 decimal places):

$$K = \begin{bmatrix} 3.1623 & 2.7065 \end{bmatrix}$$

Note that use of the MATLAB `lqr`() function results in exactly the same results

Kalman-Bucy Filter: Now let's consider the estimator. Again, since we are considering the steady-state case, we can look at the algebraic Riccati equation for the Kalman-Bucy filter. The gain is $L = P^e H^T R_y^{-1}$, where R_y is the covariance of the measurement noise, $\omega(t)$, and P^e is the solution of the algebraic Riccati equation:

$$0 = AP^e + P^e A^T + GQ_x G^T - P^e H^T R_y^{-1} H P^e \qquad (10)$$

where the matrix $G = \begin{bmatrix} 0 & (1/I) \end{bmatrix}^T$, results from the 1^{st}-order formulation of the dynamics in (2):

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ (1/I) \end{bmatrix} u + \begin{bmatrix} 0 \\ (1/I) \end{bmatrix} \eta(t) \triangleq Ax + Bu + G\eta(t) \qquad (11)$$

Substituting into (10) yields:

$$
\begin{aligned}
0 \;=\; & \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\
& + \begin{bmatrix} 0 \\ (1/I) \end{bmatrix} (0.001) \begin{bmatrix} 0 & (1/I) \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0.5)^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}
\end{aligned}
$$

This equation leads to the the following algebraic equations for the components of P^e:

$$
\begin{aligned}
0 &= 2p_{12} - 2p_{11}^2 \\
0 &= p_{22} - 2p_{11} p_{12} \\
0 &= \frac{0.001}{I^2} - 2p_{12}^2
\end{aligned}
$$

which yields

$$p_{12} = \sqrt{\frac{0.001}{2I^2}} \qquad\qquad p_{11} = \sqrt{p_{12}}\ .$$

Hence, the steady-state estimator gain matrix is:

$$L = PH^T R_y^{-1} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (0.5)^{-1} = 2 \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = 2 \begin{bmatrix} (0.001/(2I^2))^{\frac{1}{4}} \\ (0.001/(2I^2))^{\frac{1}{2}} \end{bmatrix}\ .$$

For the case of $I = 1$, the gain matrix is $L = \begin{bmatrix} 0.2991 & 0.0447 \end{bmatrix}^T$. Note that also solving for the gain matrix using the MATLAB `kalman`() function yields exactly the same result.

Part(b): Compute the closed loop poles of the controller, and also the closed loop poles of the estimator.

Solution:
The closed loop poles of the controller are found from $(A - BK)$:

$$(A - BK) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ (1/I) \end{bmatrix} \begin{bmatrix} I\sqrt{10} & I\sqrt{2I\sqrt{10}+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{10} & \sqrt{2I\sqrt{10}+1} \end{bmatrix}$$

For the case of $I = 1$, the eigenvalues of $(A - BK)$ are $-1.3532 \pm 1.1537i$.

The closed loop poles of the estimator are found from $(A - LH)$:

$$(A - LH) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - 2 \begin{bmatrix} (0.001/(2I^2))^{\frac{1}{4}} \\ (0.001/(2I^2))^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 2(0.001/(2I^2))^{\frac{1}{4}} & 1 \\ (0.001/(2I^2))^{\frac{1}{2}} & 0 \end{bmatrix}.$$

For the case of $I = 1$, the eigenvalues of $(A - LH)$ are $-0.1495 \pm 0.1495i$.