Chapter 2

The Configuration Space of a Rigid Body

This chapter considers the problem of a freely moving rigid body, $\mathcal{B}$, surrounded by stationary rigid bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The stationary bodies represent fingertips, fixturing elements, or terrain segments supporting $\mathcal{B}$ against gravity. The body $\mathcal{B}$ represents the object begin grasped, a workpiece, or the rigidified multi-legged vehicle. This chapter introduces the notion of the rigid-body configuration space, or *c-space*, which is essential for analyzing the mobility and stability of $\mathcal{B}$ with respect to its surrounding bodies. The chapter begins with a parametrization of $\mathcal{B}$'s c-space in terms of hybrid coordinates. Configuration space obstacles (c-obstacles) are then introduced, and several of their properties are described. The chapter proceeds to describe the first and second-order geometry of the c-space obstacles, as this geometry plays a key role in subsequent chapters. Finally, the notion of generalized forces or wrenches is introduced in the context of configuration space.

2.1 The Notion of Configuration Space

We assume that $\mathcal{B}$ is a *rigid body*, which we model as a set of points positioned within an ambient *worksapce*, which is assumed to be in an n-dimensional Euclidean space, $I\!R^n$, where $n=2$ or 3. Rigidity imples that the distance between $\mathcal{B}$'s constituent points is fixed. $\mathcal{B}$'s *configuration* specifies the stationary state of the object in the workspace. Equivalently, the position of each of $\mathcal{B}$'s constituent points can be determined from its configuration. The specification of $\mathcal{B}$'s configuration requires a selection of two frames, depicted in Figure 2.1. The first is a fixed *world frame*, denoted $\mathcal{F}_W$, which establishes a coordinate system for the workspace in which $\mathcal{B}$ moves. The second is a *body frame*, denoted $\mathcal{F}_B$, which is rigidly attached to $\mathcal{B}$. The configuration of $\mathcal{B}$ can be specified by a vector $d \in I\!R^n$ describing the position of $\mathcal{F}_B$'s origin with respect to the origin of $\mathcal{F}_W$, and a *rotation matrix*, $R \in I\!R^{n\times n}$, whose columns describe the relative orientation of the axes of $\mathcal{F}_B$ with respect to those of $\mathcal{F}_W$. The collection of $n\times n$ orientation matrices forms a group under matrix multiplication, termed the *special orthogonal group*, and denoted by the symbol $SO(n)$.

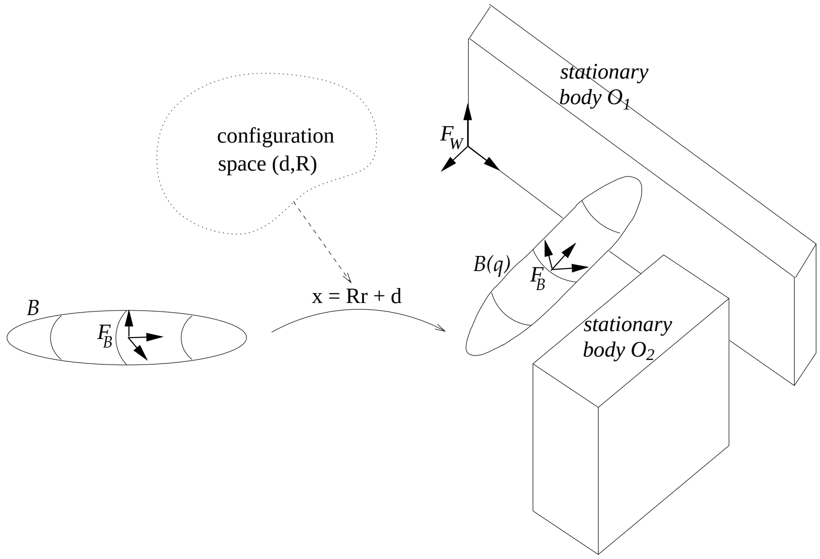

Figure 2.1: The physical geometry underlying the c-space representation of a 3D body $\mathcal{B}$. Think of $\mathcal{B}$'s configuration as a *placement* of $\mathcal{B}$ in its workspace.

Characterization of $SO(n)$. The special orthogonal group of $n \times n$ orientation matrices is given by

$$SO(n) = \left\{ R \in I\!R^{n\times n} : R^T R = I \quad \text{and} \quad \det(R) = 1 \right\},$$

where I is an $n \times n$ identity matrix.

This characterization of $SO(n)$ provides two important insights. First, every rotation matrix acts on vectors $v \in I\!R^n$ so as to preserve their length, since $\|Rv\| = (v^T R^T R v^T)^{1/2} = \|v\|$. Second, $SO(n)$ is a compact smooth manifold of dimension $\frac{1}{2}n(n-1)$ in the space $I\!R^{n\times n}$. In particular, $SO(2)$ is a one-dimensional loop in the space of 2×2 matrices, while $SO(3)$ is a compact three-dimensional manifold in the space of 3×3 matrices.

Definition 1 (Configuration Space)**.** *The configuration space of $\mathcal{B}$, denoted $\mathcal{C}$, is the smooth manifold $\mathcal{C} = I\!R^n \times SO(n)$, consisting of pairs (d, R) such that $d \in I\!R^n$ and $R \in SO(n)$.*

The dimension of $\mathcal{C}$ is the sum: $m = n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$, giving $m = 3$ when $\mathcal{B}$ is a 2-dimensional (2D) body and $m = 6$ when $\mathcal{B}$ is a 3-dimensional (3D) body. We now introduce a parametrization of $\mathcal{C}$ in terms of *hybrid coordinates* [7]. This parametrization allows us to locally represent $\mathcal{C}$ as a Euclidean space $I\!R^m$, with some periodicity rules for the coordinates representing the orientation matrices.

We first introduce coordinates for $SO(n)$. The group $SO(n)$ is an important instance of a *Lie group.*[1] A standard means for parametrizing Lie groups is via *exponential coordinates*:

$$R(\boldsymbol{\theta}) \;=\; e^{[\boldsymbol{\theta}\times]}$$

where the matrix exponential can be formally defined via the series: $\exp(A) = I + A + \frac{1}{2!}A^2 + \cdots$, and where $[\boldsymbol{\theta}\times]$ is a skew-symmetric matrix[2].

[1] Lie groups are matrix groups possessing a smooth manifold structure.
[2] These skew-symmetric matrices form the *Lie Algebra* of the Lie group SO(3).

Exponential Coordinates for $SO(n)$. The exponential coordinate for $SO(2)$ is a scalar θ (since $SO(2)$ is a one-dimensional manifold). The skew symmetric matrix in the matrix exponential representation of $SO(2)$ is the 2×2 matrix $[\theta\times] = \theta J$ where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Consequently, the $2{\times}2$ orientation rotation matrices are globally parametrized by the formula

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \in I\!R,$$

where θ is the relative orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$, measured using the right-hand-rule (which measures angles in the counterclockwise direction around the upward-pointing normal to the plane).

For 3×3 rotation matrices in $SO(3)$, the skew symmetric matrix $[\boldsymbol{\theta}\times]$ has a physical interpretation as a *cross-product* matrix: $[\boldsymbol{\theta}\times]\vec{v} = \boldsymbol{\theta} \times \vec{v}$ for any vector $\vec{v} \in I\!R^3$. The direction of the vector $\boldsymbol{\theta}$ physically corresponds to the *axis of rotation*, and the norm of the vector, $||\boldsymbol{\theta}||$, corresponds to the angle of rotation[3] about the axis of rotation. For $SO(3)$, it can be shown that the matrix exponential formula reduces to *Rodriguez' Formula*:

$$R(\boldsymbol{\theta}) = I + \sin(\|\boldsymbol{\theta}\|)[\hat{\boldsymbol{\theta}}\times] + \big(1-\cos(\|\boldsymbol{\theta}\|)\big)[\hat{\boldsymbol{\theta}}\times]^2 \quad \boldsymbol{\theta} \in I\!R^3,$$

where I is a 3×3 identity matrix and $[\hat{\boldsymbol{\theta}}\times]$ is the cross-product matrix of $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}/\|\boldsymbol{\theta}\|$. In Rodrigez' formula $\hat{\boldsymbol{\theta}}$ and $\|\boldsymbol{\theta}\|$ are the axis and angle of rotation of $R(\boldsymbol{\theta})$, measured according to the right-hand rule.

The parametrization of $SO(2)$ is periodic in 2π, with each 2π interval parametrizing the entire $SO(2)$. The parametrization of $SO(3)$ in terms of $\boldsymbol{\theta}$ satisfies the following periodicity rule. The origin of $\boldsymbol{\theta}$-space is mapped by $R(\boldsymbol{\theta})$ to the identity matrix I. Similarly, all concentric spheres of radius $\|\boldsymbol{\theta}\| = 2\pi, 4\pi, \ldots$ are mapped to the identity matrix I. Each pair of antipodal points on the sphere of radius $\|\boldsymbol{\theta}\| = \pi$ is mapped to the same matrix R, since $R(\pi\hat{\boldsymbol{\theta}}) = R(-\pi\hat{\boldsymbol{\theta}})$ for all $\hat{\boldsymbol{\theta}}$. Similarly, antipodal points on the spheres of radius $\|\boldsymbol{\theta}\| = 3\pi, 5\pi, \ldots$ are identified. Consider now a path in $\boldsymbol{\theta}$-space from the origin to the sphere of radius π along a fixed direction $\hat{\boldsymbol{\theta}}$. This path represents a rotation of $\mathcal{B}$ about the axis of rotation $\hat{\boldsymbol{\theta}}$ by an angle which increases from zero to π. Rotation of $\mathcal{B}$ about the same axis by angles π to 2π continues on a path which moves along $-\hat{\boldsymbol{\theta}}$ back to the origin. Since $\hat{\boldsymbol{\theta}}$ can have any direction, the entire manifold $SO(3)$ is parametrized by the ball with center at the origin and radius π, with antipodal points on its bounding sphere identified.

Definition 2 (Hybrid Coordinates)**.** *When $\mathcal{B}$ is a 2D body, the* **hybrid coordinates** *for its c-space are $q = (d, \theta) \in I\!R^2 \times I\!R$. When $\mathcal{B}$ is a 3D body, the* hybrid coordinates[4] *for its c-space are $q = (d, \boldsymbol{\theta}) \in I\!R^3 \times I\!R^3$.*

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[3] *Euler's Theorem* states that every rigid body rotation is equivalent to a rotation about a fixed axis.

[4] Formally, the **hybrid coordinates** are $I\!R^n \times \mathfrak{se}(n)$, where $\mathfrak{se}(n)$ is the Lie algebra of $SO(n)$. However, since $\mathfrak{se}(n)$ is isomorphic to $I\!R^n$, $I\!R^n$ is used hereafter for simplicity.

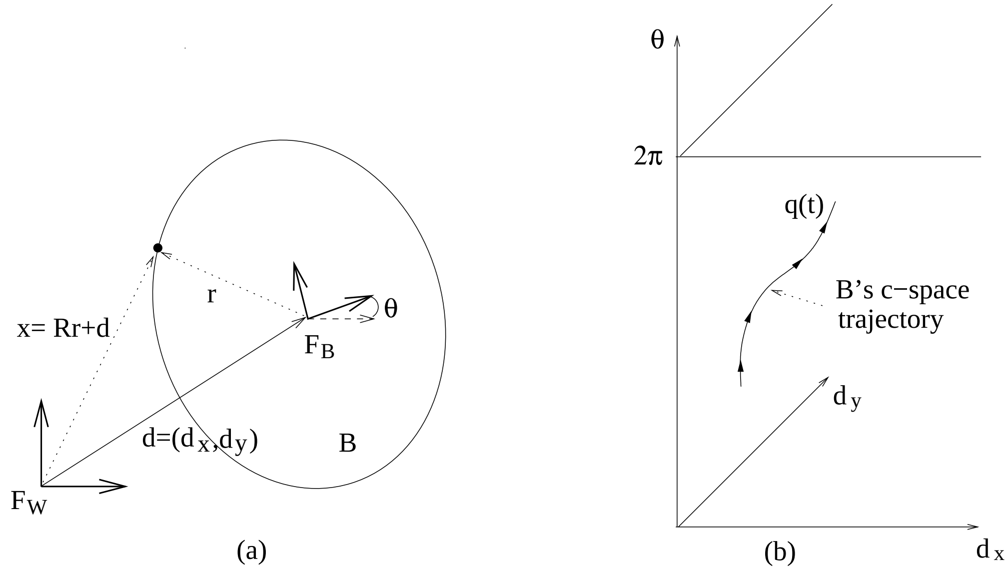

Figure 2.2: (a) Hybrid coordinates $q = (d_x, d_y, \theta)$ for $\mathcal{B}$'s c-space. (b) A c-space trajectory representing $\mathcal{B}$'s physical motion.

When $\mathcal{B}$ is a 2D body, its c-space parametrization is simply $I\!R^3$ in hybrid coordinates, partitioned into 2π layers along the θ axis (see Figure 2.2). Each 2π layer provides a full parametrization of c-space. Hence a path $q(t)$ can freely move between layers, or it can remain in a particular layer by wrapping through its bounding planes. When $\mathcal{B}$ is a 3D body its c-space parametrization is simply $I\!R^6$ in hybrid coordinates, with the $\boldsymbol{\theta}$ coordinates partitioned into a central ball and concentric shells each having a radius/thickness of π. Here, too, a path $q(t)$ can freely move between neighboring shells, or it can remain in the inner ball by wrapping through antipodal points on its bounding sphere.

To summarize, c-space allows us to model the physical motions of $\mathcal{B}$ as trajectories, $q(t)$, of a point in $\mathcal{C}$, which is parametrized by $I\!R^m$, where $m = 3$ (for rigid bodies moving in the plane) or 6 (for bodies moving in 3-dimensional Euclidean space). Before we proceed to fill this space with forbidden regions representing the stationary finger bodies, let us review the notion of a rigid-body transformation.

The rigid-body transformation. As $\mathcal{B}$ moves along a c-space trajectory $q(t)$, the position of its points with respect to the world frame $\mathcal{F}_W$ is specified as follows. Let b denote the postion of a point in $\mathcal{B}$, as seen by an observer the body frame $\mathcal{F}_B$, and let x denote the coordinates of the same point as seen by an observer in $\mathcal{F}_W$ (Figure 2.2(a)). The *rigid-body transformation*, denoted $X(q, b)$, gives the world position of $\mathcal{B}$'s points at a configuration q,

$$x = X(q,b) \triangleq \begin{cases} R(\theta)b + d & q = (d,\theta) \in I\!R^3,\ b \in \mathcal{B} \quad \text{(2D case)} \\ R(\boldsymbol{\theta})b + d & q = (d,\boldsymbol{\theta}) \in I\!R^6,\ b \in \mathcal{B} \quad \text{(3D case).} \end{cases}$$

The notation $X_b(q)$ will specify the rigid-body transformation such that the point $b \in \mathcal{B}$ is held fixed. In this case $X_b(q)$ gives the world position of the fixed point b as a function of q.

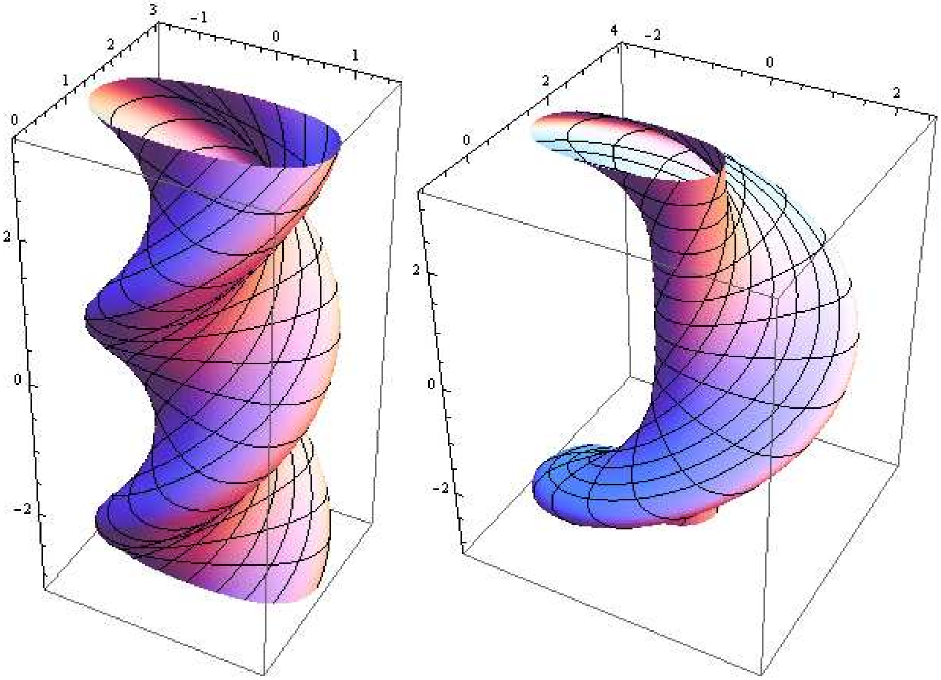

Figure 2.3: The c-obstacle induced by a stationary disc, shown for two choices of $\mathcal{F}_B$'s origin: (a) at the ellipse's center, and (b) at the ellipse's tip.

2.2 Configuration Space Obstacles

The rigid stationary bodies $\mathcal{O}_1 \dots \mathcal{O}_k$ form obstacles which constrain the possible motions of $\mathcal{B}$. Since it is physically impossible for two different rigid bodies to occupy the same space, the stationary bodies induce forbidden regions in $\mathcal{B}$'s c-space, called *c-obstacles*. Let $\mathcal{B}(q)$ denote the set of physical points occupied by $\mathcal{B}$ when it is at a configuration q, and let $\mathcal{O}$ be one of the stationary bodies, which is also modeled as a set of points. The *c-obstacle* induced by $\mathcal{O}$, denoted $\mathcal{CO}$, is the set of configurations q at which the set $\mathcal{B}(q)$ intersects the set $\mathcal{O}$,

$$\mathcal{CO} \stackrel{\triangle}{=} \{q \in \mathcal{C} : \mathcal{B}(q) \cap \mathcal{O} \neq \emptyset\} \quad \text{where } m = 3 \text{ or } 6.$$

When $\mathcal{B}$ is an n-dimensional body, the c-obstacle $\mathcal{CO}$ is an m-dimensional set in the ambient m-dimensional c-space, even when $\mathcal{O}$ is a point obstacle. The boundary of $\mathcal{CO}$ is an $(m-1)$-dimensional set, consisting of configurations at which $\mathcal{B}$ touches $\mathcal{O}$ from the outside. A curve on $\mathcal{CO}$'s boundary represents a motion of $\mathcal{B}$ which maintains continuous contact with $\mathcal{O}$. In planar environments the boundary of $\mathcal{CO}$ can be conceptually constructed as follows. First fix the orientation of $\mathcal{B}$ to a particular value θ. Then move $\mathcal{B}$ along the perimeter of $\mathcal{O}$ with this fixed orientation, making sure that $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$. The trace of $\mathcal{B}$'s origin during this circumnavigation forms a closed curve which is precisely the boundary of the *fixed-orientation slice* of $\mathcal{CO}$. When this process is repeated for all θ, the resulting set of curves forms a representation of the c-obstacle boundary.

Example 1. Figure 2.3(a) shows an ellipse $\mathcal{B}$ moving in a planar environment populated by a stationary circular disc $\mathcal{O}$. The c-obstacle induced by $\mathcal{O}$ is depicted in Figure 2.3(b) for two choices of $\mathcal{F}_B$'s origin, at the ellipse's center and at the of the ellipse's major axis. While the two c-obstacles differ in their geometric shape, they are topologically equivalent. This observation holds true under any choice of $\mathcal{F}_W$ and $\mathcal{F}_B$.

The c-obstacle distance function. An analytic description of the c-obstacle can be constructed as follows. Let $\mathsf{dst}(x, \mathcal{O})$ denote the minimal distance of a point x from a fixed set $\mathcal{O}$, given by

$$\mathsf{dst}(x, \mathcal{O}) = \min_{y \in \mathcal{O}} \{\|x - y\|\} \ .$$

The minimal distance between $\mathcal{B}(q)$ and $\mathcal{O}$, denoted $d(q)$, is defined by

$$d(q) \triangleq \min_{x \in \mathcal{B}(q)} \{\mathsf{dst}(x, \mathcal{O})\} = \min_{b \in \mathcal{B}} \left\{\mathsf{dst}\big(X(q, b), \mathcal{O}\big)\right\},$$

where $x = X(q, b)$ is the rigid-body transformation of the point $b \in \mathcal{B}$ when $\mathcal{B}$ lies at configuration q. Note that $d(q)$ is strictly positive for all q lying outside $\mathcal{CO}$, and is identically zero for any q lying inside $\mathcal{CO}$. Hence the c-obstacle $\mathcal{CO}$ is described by the inequality,

$$\mathcal{CO} = \{q \in \mathcal{C} : d(q) \leq 0\}. \tag{2.1}$$

One can equivalently write $\mathcal{CO} = \{q \in \mathcal{C} : d(q) = 0\}$, but formulation (2.1) anticipates later chapters where c-space is used to analyze the motions of a quasi-rigid body.

A detailed discussion of c-obstacles can be found in textbooks dedicated to robot motion planning [1, 2, 4, 5]. The following list summarizes some of their key properties [5].

1. **Compactness and connectivity propagate.** When rigid body $\mathcal{B}$ is a compact and path connected [6] set, any compact and path connected obstacle $\mathcal{O}$ induces a compact and path connected c-obstacle $\mathcal{CO}$.

2. **Union propagates.** When an obstacle $\mathcal{O}$ is a union of two sets, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, its c-obstacle is a union of the c-obstacles corresponding to the individual sets, $\mathcal{CO} = \mathcal{CO}_1 \cup \mathcal{CO}_2$.

3. **Convexity propagates.** Recall that a set $\mathcal{S} \subseteq I\!R^n$ is *convex* if every pair of points in $\mathcal{S}$ can be connected by a line segment lying wholly in $\mathcal{S}$. When $\mathcal{O}$ and $\mathcal{B}$ are convex bodies, each fixed-orientation slice of $\mathcal{CO}$ is a convex set.

4. **Polygonality propagates.** When $\mathcal{B}$ and $\mathcal{O}$ are polygonal bodies, each fixed-orientation slice of $\mathcal{CO}$ is a two-dimensional polygonal set. When $\mathcal{B}$ and $\mathcal{O}$ are polyhedral bodies, each fixed-orientation slice of $\mathcal{CO}$ is a three-dimensional polyhedral set.

[5]The term "propagate in this list implies that the property in the n-dimensional Euclidean workspace propagates, or is conserved, under the mapping to configuration space.n

[6]A set, $\mathcal{S}$, is set to be path-connected if for any two points $x, y \in \mathcal{S}$, there exists a continuous path lying within $\mathcal{S}$ which connects x and y. Connected set in $\mathbb{R}^n$ are necessarily path-connected.

2.2.1 Construction the c-obstacle boundary

Planar polygonal object. A popular method for computing the explicit shape of the c-obstacles for the case of convex planar polygonal bodies is known as the *star algorithm*. The method assumes that $\mathcal{B}$ and $\mathcal{O}$ are both convex polygons. In this case, each fixed-θ slice of $\mathcal{CO}$, denoted $\mathcal{CO}|_\theta$, is also a convex polygon. The vertices of $\mathcal{CO}|_\theta$ correspond to configurations at which a vertex of $\mathcal{B}$ (having a fixed orientation θ) touches a vertex of $\mathcal{O}$, such that the bodies' interiors are disjoint. The vertices on the boundary of $\mathcal{CO}|_\theta$ can be computed by a simple algorithm which merges the vertices of $\mathcal{B}$ and $\mathcal{O}$ on a common unit circle known as the *star circle* [2, 4].

Planar convex smooth bodies. When $\mathcal{B}$ is a smooth convex body and $\mathcal{O}$ is a circular disc, one can explicitly parametrize the boundary of $\mathcal{CO}$ as follows. First note that as $\mathcal{B}$ traces the perimeter of $\mathcal{O}$ with a fixed orientation, the contact point monotonically traces the entire perimeter of $\mathcal{B}$. Also note that the operation where $\mathcal{B}$ traces, with a fixed orientation, the perimeter of $\mathcal{O}$ in $\mathcal{F}_W$ is equivalent to an operation where $\mathcal{O}$ traces the perimeter of the stationary $\mathcal{B}$ in $\mathcal{F}_B$. Based on these observations, let $\beta(s)$ for $s \in \mathbb{R}^+$ be an arc-length parametrization[7] of $\mathcal{B}$'s perimeter in $\mathcal{F}_B$, which implies that the tangent $\beta'(s)$ is a unit vector. Assume that the boundary is parameterized in a clockwise fashion so that $J\beta'(s)$ is the unit outward normal to $\mathcal{B}$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let r be the radius of disc $\mathcal{O}$, and let x_0 be the position of its center in $\mathcal{F}_W$. Then during a motion of $\mathcal{O}$ along $\mathcal{B}$'s perimeter, the curve traced by $\mathcal{O}$'s center in $\mathcal{F}_B$ is: $\beta(s) + rJ\beta'(s)$ for $s \in \mathbb{R}$. Based on a simple calculation (see Exercise 8), the curve traced by $\mathcal{B}$'s origin in $\mathcal{F}_W$ is: $d(s,\theta) = x_0 - R(\theta)\big(\beta(s) + rJ\beta'(s)\big)$, where $R(\theta)$ is $\mathcal{B}$'s fixed orientation matrix. When θ varies freely in $\mathbb{R}$, the function $\varphi(s,\theta) = \big(d(s,\theta),\theta\big) : \mathbb{R}^2 \to \mathbb{R}^3$ provides a parametrization of $\mathcal{CO}$'s boundary in term of s and θ. The c-obstacles depicted in Figure 2.3 were generated using this technique.

Example 2. *Assume that $\mathcal{B}$ is a convex smooth body whose boundary is arc-length parametrized (in a clockwise fashion) by the function $\beta(s)$. Consider an elliptical obstacle, described by: $(x - x_0)^T P(x - x_0) \leq 1$ where $P > 0$. At the contact point $x(s)$: $P(x(s) - x_0) = -\lambda R(\theta_0)J\beta'(s)$ for some $\lambda > 0$. Multiplying both sides by $P^{-1/2}$ gives: $P^{1/2}(x(s) - x_0) = -\lambda P^{-1/2}R(\theta_0)J\beta'(s)$. Taking the norm of both sides gives:*

$$1 = (x(s) - x_0)^T P(x(s) - x_0) = \lambda \|P^{-1/2}R(\theta_0)J\beta'(s)\| \Rightarrow \lambda(s) = \frac{1}{\|P^{-1/2}R(\theta_0)J\beta'(s)\|}.$$

Substituting for $\lambda(s)$ in the contact-normals equation gives

$$P(x(s) - x_0) = -\lambda(s)R(\theta_0)J\beta'(s) \Rightarrow x(s) = x_0 - \lambda(s)P^{-1}R(\theta_0)J\beta'(s).$$

On the other hand, $x(s) = R(\theta_0)b(s) + d(s)$. Substituting for $x(s)$ and solving for $d(s)$ gives

$$\begin{aligned} d(s,\theta) &= x(s) - R(\theta)b(s) = x_0 - \lambda(s)P^{-1}R(\theta)J\beta'(s) - R(\theta)b(s) \\ &= x_0 - R(\theta)\left(b(s) + \lambda(s)P^{-1}J\beta'(s)\right), \end{aligned}$$

where θ is now freely varying in $\mathbb{R}$. Note that $b(s) + \lambda(s)P^{-1}J\beta'(s)$ is the curve traced by $\mathcal{O}$'s center in $\mathcal{F}_B$.

[7] An appendix to this chapter briefly reviews the differential geometry of curves and surfaces.

The c-obstacle boundary is generally a piecewise smooth surface in the 2D case. For instance, when $\mathcal{B}$ is a convex polygon and $\mathcal{O}$ is a disc, $\mathcal{CO}$'s boundary consists of two types of smooth two-dimensional "patches" meeting along one-dimensional curves. An edge-patch generated by an edge of $\mathcal{B}$ sliding on $\mathcal{O}$, and a vertex-patch generated by a vertex of $\mathcal{B}$ sliding on $\mathcal{O}$. The boundary of $\mathcal{CO}$ is locally smooth at any configuration at which $\mathcal{B}$ touches $\mathcal{O}$ at a single point, such that the two bodies are smooth in the vicinity of the contact. In particular, the entire boundary of $\mathcal{CO}$ is smooth when $\mathcal{B}$ and $\mathcal{O}$ are smooth convex bodies (see exercise). Similar observations hold for the five-dimensional boundary of $\mathcal{CO}$ in the 3D case.

2.3 The C-Obstacles 1'st and 2'nd-Order Geometry

When $\mathcal{B}$ is contacted by stationary finger bodies $\mathcal{O}_1, \ldots, \mathcal{O}_k$, its configuration q lies on the boundary of each c-obstacle $\mathcal{CO}_i$ for $i = 1, \ldots, k$. We shall see in Chapter 4 that the free motions of $\mathcal{B}$ are determined in this case by the first and second-order geometry of the c-obstacle boundaries i.e., by the c-obstacles' normal and curvature. Let us now focus on a particular stationary body $\mathcal{O}$, and derive formulas for the normal and curvature of its c-obstacle boundary, denoted $\mathsf{bdy}(\mathcal{CO})$. We shall assume that $\mathcal{B}$ touches $\mathcal{O}$ at a single point, such that the two bodies have smooth boundaries in the vicinity of the contact. We first obtain a formula for the c-obstacle normal, then obtain a formula for its curvature.

2.3.1 The C-Obstacle Normal

By construction $\mathcal{CO} = \{q \in \mathcal{C} : d(q) \leq 0\}$. If the distance function $d(q)$ were differentiable at $q \in \mathsf{bdy}(\mathcal{CO})$, its gradient $\nabla d(q)$ would be collinear with the c-obstacle outward normal at q. But $d(q)$ is identically zero inside $\mathcal{CO}$ and is monotonically increasing away from $\mathcal{CO}$, implying that it is *non-differentiable* at $q \in \mathsf{bdy}(\mathcal{CO})$. However, because $d(q)$ is Lipschitz continuous (the notion of Lipschitz continuity and other relevant aspects of non-smooth analysis are reviewed in Appendix B), its differential properties can be analyzed. Lipschitz continuous functions are automatically piecewise smooth, and they possess a generalized gradient at points were the function is non-differentiable. The *generalized gradient* of a Lipschitz continuous function f at x, denoted $\partial f(x)$, is the convex combination of the gradients $\nabla f(y)$ for all y in an arbitrarily small neighborhood of x (see appendix). In particular, $\partial f(x)$ reduces to $\nabla f(x)$ at points where f is differentiable. Let us now compute $\partial d(q)$ and see how it determines the c-obstacle normal.

The c-obstacle distance function is the minimum over a parametrized family of functions, $d(q) = \min\{\mathsf{dst}\big(X(q,b), \mathcal{O}\big)\}$ such that b varies in $\mathcal{B}$. In order to emphasize that only q is a free variable, let us write $d(q) = \min_{b \in \mathcal{B}}\{\mathsf{dst}\big(X_b(q), \mathcal{O}\big)\}$. Two basic results from non-smooth analysis are needed to compute $\partial d(q)$. The first concerns the minimum over a parametrized family of functions.

Theorem 1 (Generalized Gradient of Pointwise Minimum)**.** *Let $f_t(x)$ for $t \in \mathcal{T}$ be a parametrized family of functions such that $f_t(x)$ is Lipschitz continuous in x for each $t \in \mathcal{T}$. Let $F(x) = \min_{t \in \mathcal{T}}\{f_t(x)\}$. Then $F(x)$ is also Lipschitz continuous in x. When the minimum at x is* *attained by a discrete set of functions, $F(x) = f_{t_1}(x) = \cdots = f_{t_N}(x)$, the generalized gradient of $F(x)$, denoted by $\partial F(x)$, is given by*

$$\partial F(x) = \sum_{j=1}^{N} \lambda_j \partial f_{t_j}(x) \quad 0 \leq \lambda_j \leq 1 \textit{ for } j = 1 \ldots N \textit{ and } \textstyle\sum_{j=1}^{N} \lambda_j = 1,$$

where $\partial f_{t_j}(x)$ is the generalized gradient of $f_{t_j}(x)$ for $j = 1 \ldots N$.

Note that $\partial F(x)$ is a convex combination of $\partial f_{t_j}(x)$ for $j = 1 \ldots N$. In order to apply the theorem to the family of functions $\mathsf{dst}\big(X_b(q), \mathcal{O}\big)$ for $b \in \mathcal{B}$, we must verify that each of these functions is Lipschitz continuous in q. The rigid-body transformation $X_b(q)$ is smooth in q and therefore Lipschitz continuous in q. The minimal distance function, $\mathsf{dst}(x, \mathcal{O})$, is shown in the appendix to be Lipschitz continuous in x. Since Lipschitz continuity is preserved under function composition (see appendix), each function $\mathsf{dst}(X_b(q), \mathcal{O})$ is Lipschitz continuous. Hence we may apply the theorem to the computation of $\partial d(q)$. Let $b_0 \in \mathcal{B}$ be the contact point of $\mathcal{B}(q)$ with $\mathcal{O}$. Then the minimum over the functions $\mathsf{dst}(X_b(q), \mathcal{O})$ such that b varies in $\mathcal{B}$ is attained by the single function $\mathsf{dst}(X_{b_0}(q), \mathcal{O})$. Based on the theorem, the generalized gradient of $d(q)$ is given by

$$\partial d(q) = \partial \mathsf{dst}\big(X_{b_0}(q), \mathcal{O}\big).$$

The function $\mathsf{dst}\big(X_{b_0}(q), \mathcal{O}\big)$ is a composition of $\mathsf{dst}(x, \mathcal{O})$ with $x = X_{b_0}(q)$. The generalized gradient of such a composition can be computed with the following *generalized chain rule.*

Theorem 2 (Generalized Chain Rule)**.** *Let $g(y) : I\!R^n \to I\!R$ be Lipschitz continuous and let $h(x) : I\!R^m \to I\!R^n$ be a differentiable function. Then the function composition $f(x) = g(h(x))$ is Lipschitz continuous, and at points where the Jacobian $Dh(x)$ has full-rank the generalized gradient of f is given by*

$$\partial f(x) = \partial g\big(h(x)\big) \cdot Dh(x),$$

where $\partial g\big(h(x)\big)$ is the generalized gradient of $g(y)$ evaluated at $y = h(x)$ (i.e. $\partial f(x) = \{w \cdot Dh(x) : w \in \partial g(h(x))\}$).

Based on the generalized chain rule, $\partial \mathsf{dst}\big(X_{b_0}(q), \mathcal{O}\big) = \partial \mathsf{dst}(x_0, \mathcal{O}) \cdot DX_{b_0}(q)$, where $x_0 = X_{b_0}(q)$ is the world position of the contact point b_0. The Jacobian $DX_{b_0}(q)$ is given by the formula (see exercise),

$$DX_{b_0}(q) = \begin{cases} [I \;\; JR(\theta)b_0]_{2\times 3} & q = (d, \theta) \in I\!R^3 \text{ (2D case)} \\ [I \;\; [(R(\boldsymbol{\theta})b_0)\times]\;]_{3\times 6} & q = (d, \boldsymbol{\theta}) \in I\!R^6 \text{ (3D case)}, \end{cases}$$

where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, I is a 2×2 identity matrix in the 2D case and a 3×3 identity matrix in the 3D case, and $[R(\boldsymbol{\theta})b_0 \times]$ is a 3×3 cross-product matrix. As shown in the appendix, the generalized gradient of $\mathsf{dst}(x, \mathcal{O})$ at x_0 is given by

$$\partial \mathsf{dst}(x_0, \mathcal{O}) = s \cdot n(x_0) \quad \text{for } 0 \leq s \leq 1,$$

where $n(x_0)$ is the outward unit normal to $\mathcal{O}$ at x_0. Substituting for $DX_{b_0}(q)$ and $\partial\mathsf{dst}(x_0,\mathcal{O})$ in $\partial d(q) = \partial\mathsf{dst}(x_0,\mathcal{O}) \cdot DX_{b_0}(q)$, and then taking the transpose (to represent $\partial d(q)$ as a column vector), yields:

$$\partial d(q) = sDX_{b_0}^T(q)n(x_0) = \begin{cases} s\begin{pmatrix} n(x_0) \\ n(x_0)\cdot JR(\theta)b_0 \end{pmatrix} & 0\leq s\leq 1 \text{ (2D case)} \\ s\begin{pmatrix} n(x_0) \\ R(\boldsymbol{\theta})b_0 \times n(x_0) \end{pmatrix} & 0\leq s\leq 1 \text{ (3D case)}. \end{cases} \tag{2.2}$$

The generalized gradient of $d(q)$ at $q\in\mathsf{bdy}(\mathcal{CO})$ is thus a line segment with a base point at q. Moreover, $\partial d(q)$ points outward with respect to $\mathcal{CO}$, since $d(q)$ is monotonically increasing away from $\mathcal{CO}$.

C-obstacle Normal. Let us see why the line segment $\partial d(q)$ is normal to the boundary of the the c-obstacle at q. Let $\alpha(t)$ for $t\in I\!R$ be any c-space trajectory lying in $\mathsf{bdy}(\mathcal{CO})$, such that $\alpha(0) = q$. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $\alpha(t)$, $d(\alpha(t)) = 0$ along this motion. By the generalized chain rule $\frac{d}{dt}d(\alpha(t)) = \partial d(\alpha(t)) \cdot \frac{d}{dt}\alpha(t) = 0$ for $t \in I\!R$. Since $\frac{d}{dt}\big|_{t=0}\alpha(t)$ is an arbitrary tangent vector to $\mathcal{CO}$'s boundary at q, the line segment $\partial d(q)$ is perpendicular to the tangents to $\mathcal{CO}$'s boundary at q. Let $\eta(q)$ denote the endpoint of this line segment, obtained by substituting $s = 1$ in (2.2).

$$\eta(q) = DX_{b_0}^T(q)n(x_0) = \begin{cases} \begin{pmatrix} n(x_0) \\ n(x_0)\cdot JR(\theta)b_0 \end{pmatrix} & q=(d,\theta) \in I\!R^3 \text{ (2D case)} \\ \begin{pmatrix} n(x_0) \\ R(\boldsymbol{\theta})b_0 \times n(x) \end{pmatrix} & q=(d,\boldsymbol{\theta}) \in I\!R^6 \text{ (3D case)}, \end{cases} \tag{2.3}$$

where b_0 is $\mathcal{B}$'s contact point with $\mathcal{O}$, and $n(x_0)$ is the outward unit normal to the boundary of $\mathcal{O}$ at $x_0 = X_{b_0}(q)$. We shall see in Section 2.3 that $\eta(q)$ can be interpreted as the generalized force, or *wrench*, generated by a unit-magnitude normal force acting on $\mathcal{B}$ at x_0. The vanishing of the product $\partial d(\alpha(t)) \cdot \frac{d}{dt}\alpha(t)$ reflects the physical fact that a normal contact force does no work along any contact preserving motion of $\mathcal{B}$ as it slides along $\mathcal{O}$'s boundary.

Example: Let us verify the formula for $\eta(q)$ using the parametrization $\varphi(s,\theta)$ for $\mathcal{CO}$'s boundary, associated with an elliptical body $\mathcal{B}$ and a disc finger $\mathcal{O}$. It can verified (see exercises) that the tangent vectors $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial\theta}\varphi(s,\theta)$ are linearly independent and therefore span the tangent plane to $\mathcal{CO}$ at $\varphi(s,\theta)$. The cross-product of the two tangent vectors should therefore be collinear with $\eta(q)$. A straightforward calculation yields

$$\frac{\partial}{\partial s}\varphi(s,\theta) \times \frac{\partial}{\partial\theta}\varphi(s,\theta) = \begin{pmatrix} -JR(\theta)\beta'(s) \\ (-JR(\theta)\beta'(s)) \cdot JR(\theta)\beta(s) \end{pmatrix} = \begin{pmatrix} n(x) \\ n(x)\cdot JR(\theta)b \end{pmatrix},$$

where $b = \beta(s)$, $x = R(\theta)\beta(s) + d(s,\theta)$, and $n(x) = -JR(\theta)\beta'(s)$ (since $JR(\theta) = R(\theta)J$, and $J\beta'(s)$ is the outward unit normal to $\mathcal{B}$ at $\beta(s)$). We see that the c-obstacle normal computed from $\varphi(s,\theta)$ matches the generic formula for $\eta(q)$.

2.3.2 The C-Obstacle Curvature

The c-obstacle curvature depends on the curvature of the contacting bodies as well as on lower-order geometric properties of the contacting bodies. We shall derive the c-obstacle curvature formula for the $2D$ case, where $\mathcal{CO}$'s boundary is a surface in $I\!R^3$. Let us first introduce notation for the curvature of the contacting bodies, starting with the stationary body $\mathcal{O}$. Recall that $n(x)$ is the outward unit normal to $\mathcal{O}$ at points $x \in \mathsf{bdy}(\mathcal{O})$. Let $x(t)$ be a curve on $\mathcal{O}$'s boundary such that $x(0)=x$ and $\frac{d}{dt}\big|_{t=0} x(t)=\dot{x}$. The curvature of $\mathcal{O}$ at x, denoted $\kappa_{\mathcal{O}}(x)$, is a signed scalar measuring the change in $n(x)$ along $x(t)$,

$$\frac{d}{dt}\bigg|_{t=0} n(x(t)) = \kappa_{\mathcal{O}}(x)\dot{x}.$$

Note that the change in $n(x)$ is tangent to $\mathcal{O}$'s boundary at x. The sign of $\kappa_{\mathcal{O}}(x)$ is positive when $\mathcal{O}$ is convex at x, negative when $\mathcal{O}$ is concave at x, and zero when $\mathcal{O}$ is flat at x. The radius of curvature of $\mathcal{O}$ at x, denoted $r_{\mathcal{O}}(x)$, is the reciprocal of the curvature, $r_{\mathcal{O}}(x) = 1/\kappa_{\mathcal{O}}(x)$. The circle of curvature at x is the circle tangent to $\mathcal{O}$'s boundary at x with radius $|r_{\mathcal{O}}(x)|$. It forms the boundary's second-order approximation at x. The curvature of $\mathcal{B}$ is similarly defined with respect to its body frame. The curvature of $\mathcal{B}$ at $b \in \mathsf{bdy}(\mathcal{B})$ is the signed scalar $\kappa_{\mathcal{B}}(b)$, and its radius of curvature is $r_{\mathcal{B}}(b) = 1/\kappa_{\mathcal{B}}(b)$. In the following discussion, the shorthand notation $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ will be used for $\kappa_{\mathcal{B}}(b)$ and $\kappa_{\mathcal{O}}(x)$.

Let $\mathcal{S}$ denote the boundary of the c-obstacle $\mathcal{CO}$, and let $T_q\mathcal{S}$ denote the tangent space to $\mathcal{S}$ at q. Recall that $\eta(q)$ is the outward normal to $\mathcal{CO}$ at $q \in \mathcal{S}$. Let $\hat{\eta}(q) = \eta(q)/\|\eta(q)\|$ be the unit normal to $\mathcal{S}$, and let $q(t)$ be a curve on $\mathcal{S}$ such that $q(0) = q$ and $\frac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. The curvature of $\mathcal{S}$ at q, denoted $\kappa(q,\dot{q})$, measures the change in $\hat{\eta}(q)$ along tangent directions $\dot{q} \in T_q\mathcal{S}$,

$$\kappa(q,\dot{q}) = \dot{q} \cdot \frac{d}{dt}\bigg|_{t=0} \hat{\eta}(q(t)) = \dot{q} \cdot D\hat{\eta}(q)\dot{q} \qquad \dot{q} \in T_q\mathcal{S}.$$

The curvature $\kappa(q,\dot{q})$ is a quadratic form in the tangent directions $\dot{q} \in T_q\mathcal{S}$. Since $T_q\mathcal{S}$ is a two-dimensional space, $D\hat{\eta}(q)$ acts as a 2×2 symmetric matrix on $T_q\mathcal{S}$. The eigenvalues and eigenvectors of $D\hat{\eta}(q)$ are the principal curvatures and directions of $\mathcal{S}$ at q. The principles curvatures (or their reciprocals, the principal radii of curvatures) are analogous to the scalar curvature of a planar curve. The quadratic surface tangent to $\mathcal{S}$ at q having the principal curvatures and directions of $\mathcal{S}$ forms the surface's second-order approximation at q.

C-obstacle Curvature. Recall that the curvature of a surface is related to the derivature of the unit normal vector to that surface. Since the normal to the c-obstacle surface, $\mathcal{S}$, is $\eta(q) = DX_b^T(q)n(x)$, the derivative of $\eta(q)$ along a trajectory $q(t)$ lying in $\mathcal{S}$ involves the contact point velocity. The contact point velocity depends on the curvature of $\mathcal{B}$ and $\mathcal{O}$ as stated in the following lemma (whose proof appears in the appendix to this chapter).

Lemma 2.3.1 (Contact Point Velocity)**.** *Let $q(t)$ be a c-space trajectory on $\mathcal{S}$, and let $x(t)$ be $\mathcal{B}$'s contact point with $\mathcal{O}$ along $q(t)$. Then the contact point velocity is given by*

$$\dot{x} = \tfrac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\,[I \;\; JR(\theta)b_c]\,\dot{q} \qquad \dot{q} = \tfrac{d}{dt}q(t),$$

where $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at $x(t)$, and b_c is $\mathcal{B}$'s center of curvature at $x(t)$ expressed in $\mathcal{F}_B$; I is a 2×2 identity matrix and $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

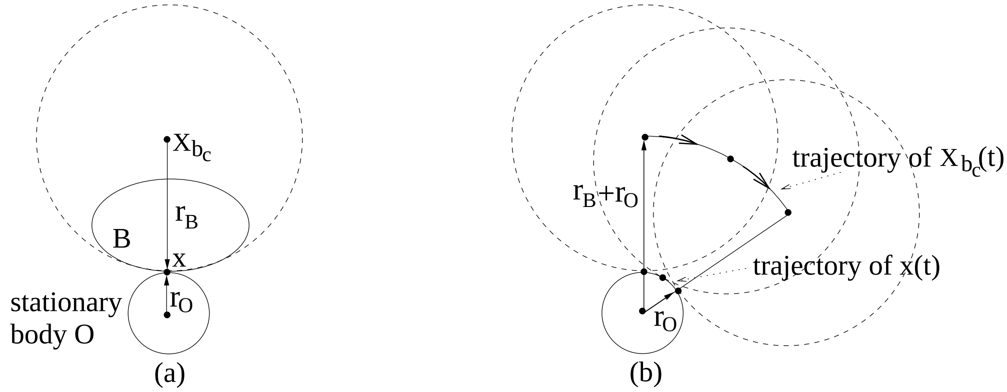

Figure 2.4: (a) The contacting bodies are replaced by their circle of curvature at x. (b) The $\mathcal{B}$-circle executes a contact preserving motion along the $\mathcal{O}$-circle.

Practically, the denominator $\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}}$ is always a non-negative quantity. For instance, when $\mathcal{O}$ is concave at x the body $\mathcal{B}$ is necessarily convex at x and $r_{\mathcal{B}} \leq |r_{\mathcal{O}}|$ (otherwise the two bodies interpenetrate). In this case $|\kappa_{\mathcal{O}}| \leq \kappa_{\mathcal{B}}$ and indeed $\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}} \geq 0$. The quantity $\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}$ is strictly positive when the bodies' second-order approximations maintain point contact at x. Since we assume a single point contact, we may as well assume that $\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}} > 0$.

The geometric interpretation of the contact velocity formula, depicted in Figure 2.4, is as follows. Let $X_{b_c}(q) = R(\theta)b_c + d$ be the world position of $\mathcal{B}$'s center of curvature at x, such that b_c is held fixed on $\mathcal{B}$. Then $\dot{X}_{b_c} = [I \ \ JR(\theta)b_c]\,\dot{q}$, and the contact velocity formula asserts that $\dot{x} = \kappa_{\mathcal{B}}/(\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}})\dot{X}_{b_c}$. In order to justify the latter formula, let $\mathcal{B}$ and $\mathcal{O}$ be replaced by their circles of curvature at x. Let the $\mathcal{B}$-circle execute any contact-preserving motion along the stationary $\mathcal{O}$-circle. The $\mathcal{B}$-circle's center, X_{b_c}, moves along a circular arc of radius $|r_{\mathcal{B}}+r_{\mathcal{O}}|$ during this motion (see Figure 2.4(b)). The circles' contact point, x, moves along a concentric circular arc of radius $|r_{\mathcal{B}}|$ during this motion. Since x and X_{b_c} lie on a common radius vector emanating from the $\mathcal{O}$-circle center, the two points move with identical angular velocities about the $\mathcal{O}$-circle center. Moreover, the two points move in the same direction when $r_{\mathcal{B}} \geq 0$. Assuming this case, let $\dot{\phi}$ be the common angular velocity of the two points. Then $\dot{x} = |r_{\mathcal{O}}|\dot{\phi}$ while $\dot{X}_{b_c} = |r_{\mathcal{B}}+r_{\mathcal{O}}|\dot{\phi}$. Substituting $\dot{\phi} = \dot{X}_{b_c}/|r_{\mathcal{B}}+r_{\mathcal{O}}|$ in the expression for $\dot{x}$ gives: $\dot{x} = |r_{\mathcal{O}}|/|r_{\mathcal{B}}+r_{\mathcal{O}}|\dot{X}_{b_c}$. Finally, $|r_{\mathcal{O}}|/|r_{\mathcal{B}}+r_{\mathcal{O}}| = \kappa_{\mathcal{B}}/(\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}})$ when $r_{\mathcal{B}} \geq 0$, giving the contact velocity formula.

Based on the contact velocity formula, the c-obstacle curvature form is as follows.

Lemma 2.3.2 (C-Obstacle Curvature)**.** *Let $\mathcal{S}$ be the boundary of $\mathcal{CO}$, let $q \in \mathcal{S}$, and let $\eta(q)$ be the normal to $\mathcal{S}$ at q. The* curvature form *of $\mathcal{S}$ at q is given by*

$$\kappa(q,\dot{q}) = \frac{1}{\|\eta(q)\|} \cdot \frac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\,\dot{q}^T \begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}} I & \kappa_{\mathcal{B}}\kappa_{\mathcal{O}} JRb_c \\ \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb_c)^T & (\kappa_{\mathcal{O}}Rb - n(x))^T(\kappa_{\mathcal{B}}Rb + n(x)) \end{bmatrix} \dot{q} \qquad \dot{q} \in T_q\mathcal{S},$$

where $x = X(q,b)$ is $\mathcal{B}$'s contact point with $\mathcal{O}$, $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at x, $n(x)$ is the contact normal at x, and b_c is $\mathcal{B}$'s center of curvature at x expressed in $\mathcal{F}_B$; I is a 2×2 identity matrix and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The following proof sketch describes how the contact velocity formula determines the c-obstacle curvature form. The complete proof appears at the end of this chapter.

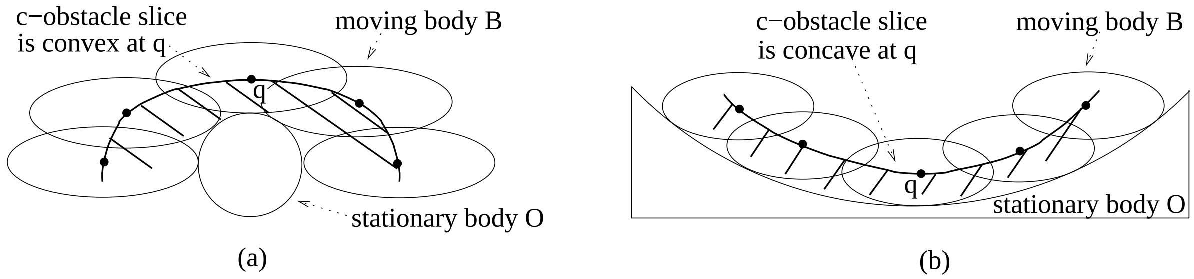

Figure 2.5: The c-obstacle slice is (a) convex when $\mathcal{B}$ and $\mathcal{O}$ are convex at x, and (b) concave when one of the two bodies is concave at x.

Proof sketch. Let $q(t)$ be a curve on $\mathcal{S}$ such that $q(0) = q$ and $\frac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. The proof first argues that the curvature form of $\mathcal{S}$ can be written in terms of $\eta(q)$ as follows

$$\kappa(q, \dot{q}) = \frac{1}{\|\eta(q)\|}\, \dot{q} \cdot \frac{d}{dt}\bigg|_{t=0} \eta(q(t)) \qquad \dot{q} \in T_q\mathcal{S}. \tag{2.4}$$

Recall that the c-obstacle normal is given by $\eta(q) = DX_b^T(q)n(x)$, where $DX_b^T(q) = [I \;\; JRb]^T$ and $n(x)$ is the unit normal to $\mathcal{O}$ at x. Thus we have to compute the derivative

$$\frac{d}{dt}\bigg|_{t=0} \eta(q(t)) = DX_b^T(q)\, \frac{d}{dt}\bigg|_{t=0} n(x(t)) + \left(\frac{d}{dt}\bigg|_{t=0} DX_b^T(q(t)) \right) n(x).$$

Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $q(t)$, the contact $x(t)$ moves along $\mathcal{O}$'s boundary. Hence $\frac{d}{dt}\big|_{t=0} n(x(t)) = \kappa_{\mathcal{O}}\dot{x}$ in the first summand. Substituting $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I \;\; JRb_c]\dot{q}$ according to the contact velocity formula gives

$$DX_b^T(q)\, \frac{d}{dt}\bigg|_{t=0} n(x(t)) = \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} I \\ (JRb)^T \end{bmatrix} [I \;\; JRb_c]\, \dot{q} = \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} I & JRb_c \\ (JRb)^T & b \cdot b_c \end{bmatrix} \dot{q},$$

where we used the identities $R^TR = J^TJ = I$. In the second summand, $\frac{d}{dt}\big|_{t=0} DX_b^T(q) = \big[O \;\; J\frac{d}{dt}\big|_{t=0}(Rb)\big]^T$, where O is a 2×2 matrix of zeroes. The proof computes the term $\frac{d}{dt}\big|_{t=0}(Rb)$ by invoking the contact velocity formula for a second time. The resulting second summand is given by

$$\big(\tfrac{d}{dt}\big|_{t=0} DX_b^T(q(t))\big)n(x) = -\tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}} \begin{bmatrix} O & \vec{0} \\ \kappa_{\mathcal{O}}n^T(x)J & \kappa_{\mathcal{B}}n^T(x)Rb_c \end{bmatrix} \dot{q}.$$

Writing the derivative $\frac{d}{dt}\big|_{t=0} \eta(q(t))$ in terms of the two summands, then pre-multiplying by $1/\|\eta(q)\|$ and $\dot{q}$, gives the formula for $\kappa(q, \dot{q})$ specified above. □

Example: Let us compute the curvature of the fixed-orientation slices of $\mathcal{CO}$, denoted $\mathcal{CO}|_\theta$. The vector tangent to $\mathcal{CO}|_\theta$ is $\dot{q} = (\dot{d}, 0)$ such that $\dot{d} \perp n(x)$. This tangent vector corresponds to an instantaneous translation of $\mathcal{B}$ along the direction tangent to $\mathcal{O}$'s boundary at x. The c-obstacle curvature along $\dot{q} = (\dot{d}, 0)$ is given by

$$\kappa\big(q, (\dot{d}, 0)\big) = \frac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}}} \|\dot{d}\|^2.$$

The coefficient preceding $\|\dot{d}\|^2$ is the curvature of the c-obstacle slice $\mathcal{CO}|_\theta$ at q. Its reciprocal is the radius of curvature of $\mathcal{CO}|_\theta$ at q,

$$\left(\frac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\right)^{-1} = r_{\mathcal{B}}(x) + r_{\mathcal{O}}(x).$$

We see that the radius of curvature of $\mathcal{CO}|_\theta$ at q is the algebraic sum of the radii of curvature of $\mathcal{B}$ and $\mathcal{O}$ at the contact. When both bodies are convex at x, $r_{\mathcal{B}}(x) \geq 0$ and $r_{\mathcal{O}}(x) \geq 0$. In this case $r_{\mathcal{B}}(x) + r_{\mathcal{O}}(x) \geq 0$, implying that $\mathcal{CO}|_\theta$ is convex at q (Figure 2.5(a)). When one of the bodies is concave at x, say $\mathcal{O}$, then $r_{\mathcal{B}}(x) \geq 0$ while $r_{\mathcal{O}}(x) < 0$. Since $|r_{\mathcal{O}}(x)| > r_{\mathcal{B}}(x)$, $r_{\mathcal{B}}(x) + r_{\mathcal{O}}(x) < 0$ in this case, implying that $\mathcal{CO}|_\theta$ is concave at q (Figure 2.5(b)).

C-Obstacle Curvature in the 3D Case. The c-obstacle curvature in the 3D case depends on the same geometric data as in the 2D case, with the bodies' surface curvatures replacing the scalar curvatures $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$. The c-obstacle formula in the 3D case is as follows (see [8, 9] for a detailed derivation). Let $\mathcal{S}$ denote the five-dimensional boundary of $\mathcal{CO}$, let $q \in \mathcal{S}$, and let $x = X(q, b)$ be $\mathcal{B}$'s contact point with $\mathcal{O}$. We assume that $\mathcal{S}$ is locally smooth at q, and denote by $T_q\mathcal{S}$ the five-dimensional tangent space of $\mathcal{S}$ at q. Let $q(t)$ be a curve on $\mathcal{S}$ such that $q(0) = q$ and $\frac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. The curvature form of $\mathcal{S}$ at q is given by $\kappa(q, \dot{q}) = \dot{q} \cdot \frac{d}{dt}\big|_{t=0} \hat{\eta}(q(t))$, where $\dot{q} \in T_q\mathcal{S}$ and $\hat{\eta}(q(t))$ is the unit normal to $\mathcal{S}$ along $q(t)$. The surface curvature of $\mathcal{B}$ and $\mathcal{O}$ at x is determined by their respective curvature forms $L_{\mathcal{B}}$ and $L_{\mathcal{O}}$. These are the linear maps which act on the tangent plane of the respective surfaces to yield the change in the surface-normal along a given tangent direction. The curvature form of $\mathcal{S}$ at q is given by

$$\kappa(q,\dot{q}) = \frac{1}{\|\eta(q)\|}\dot{q}^T \left(\begin{bmatrix} I & -[Rb\times] \\ O & [n(x)\times] \end{bmatrix}^T \begin{bmatrix} L_{\mathcal{B}}\left[L_{\mathcal{O}}+L_{\mathcal{B}}\right]^{-1} L_{\mathcal{O}} & -L_{\mathcal{O}}\left[L_{\mathcal{O}}+L_{\mathcal{B}}\right]^{-1} \\ -\left[L_{\mathcal{O}}+L_{\mathcal{B}}\right]^{-1} L_{\mathcal{O}} & -\left[L_{\mathcal{O}}+L_{\mathcal{B}}\right]^{-1} \end{bmatrix} \begin{bmatrix} I & -[Rb\times] \\ O & [n(x)\times] \end{bmatrix} + \begin{bmatrix} O & O \\ O & -\left([Rb\times]^T[n(x)\times]\right)_s \end{bmatrix} \right) \dot{q} \qquad \dot{q} \in T_q\mathcal{S},$$

where $n(x)$ is the unit normal to $\mathcal{O}$ at x, I is a 3×3 identity matrix, O is a 3×3 matrix of zeroes, and $(A)_s = \frac{1}{2}(A^T + A)$. Two comments are in order here. First, $L_{\mathcal{O}} + L_{\mathcal{B}} \geq 0$, otherwise the two surfaces would interpenetrate at the contact. In particular, $L_{\mathcal{O}} + L_{\mathcal{B}}$ is positive definite (thus invertible) in the generic case where the second-order approximations to the two surfaces maintain point contact at x. Second, the tangent vector $\dot{q} = (Rb \times n(x), n(x)) \in T_q\mathcal{S}$ is an eigenvector with a zero eigenvalue of the matrix associated with the curvature form. This tangent vector corresponds to an instantaneous rotation of $\mathcal{B}$ about its contact normal with $\mathcal{O}$. Thus, in the 3D case $\mathcal{CO}$ always possess *zero curvature* along instantaneous rotation of $\mathcal{B}$ about the contact normal with $\mathcal{O}$.

Bibliographical Notes

The notion of configuration space originated around 1980 during a collaboration between two MIT doctorate students, assigned to develop one of the first robotic assembly stations [6]. While discussing the automation of the peg-in-a-hole insertion task, they noticed that the

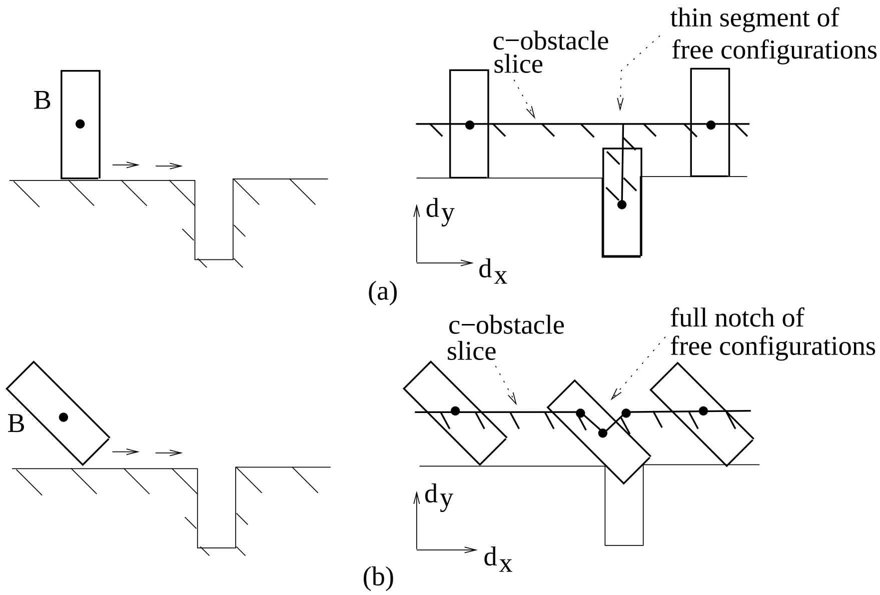

Figure 2.6: (a) At a vertical angle the peg's c-space contains only a thin segment of free configurations inside the hole. (b) At an oblique angle the peg's c-space contains a full notch of free configurations inside the hole.

best approach is to have the robot drag the peg along the surface at an oblique angle rather then at a vertical angle required for insertion (Figure 2.6). Only when the peg wedges itself into the hole, a rotational motion aligns the peg with the hole while completing the insertion task. This approach makes perfect sense if one considers the c-obstacles induced by the edges forming the hole. The θ slice of the c-obstacles at a vertical angle contains only a thin segment of collision-free configurations (Figure 2.6(a)). In contrast, the θ slice of the c-obstacles at an oblique angle contains a notch of collision-free configurations (Figure 2.6(b)). This observation led to the formulation of c-space as a model for the motions of bodies in contact, a model that has served as the basis of virtually all motion planning algorithms [1, 2, 4, 5].

Exercises

Exercise 1. Define $I\!RP^3$ in terms of lines in $I\!R^4$, then move to the sphere S^3, then to D^3. 2. Identify the non-shrinkable loop in $\boldsymbol{\theta}$-space as a loop starting at the origin, moving to the radius-π sphere, then wrapping through the antipodal point back to the origin. An attempt to shorten this loop would break it.

Exercise 2. (a) Derive a version of Rodriguez' equation for planar rigid bodies. (b) Verify that $\boldsymbol{\theta} \in I\!R^3$ ia an eigenvector of $R(\boldsymbol{\theta}) \in SO(3)$, and $\hat{x} \cdot (R(\boldsymbol{\theta})\hat{x}) = \cos(\|\boldsymbol{\theta}\|)$ for any vector $\hat{x} \in I\!R^3$ orthogonal to $\boldsymbol{\theta}$. (c) Show that Rodriguez' formula gives the 2×2 orientation matrix when $\hat{\boldsymbol{\theta}} = (0, 0, 1)$.

Exercise 3–The waiter's tray maneuver. The manifold $SO(3)$ is topologically equivalent to the three-dimensional projective space, denoted $I\!RP^3$. One way to construct $I\!RP^3$ is to take the unit three-dimensional ball (i.e. the set $\{v \in I\!R^3 : \|v\| \leq 1\}$), and identify antipodal points on its bounding sphere. The manifold $I\!RP^3$ is path connected, compact, and orientable. The one-dimensional loops in this manifold belong to two distinct classes—ones that can be shrunk to a point, and ones that wrap around a "hole" in the manifold. This non-trivial loop class can be demonstrated via the following tray maneuver. Imagine a waiter supporting a tray in a horizontal position, with elbow initially pointing downward. The waiter first rotates the tray horizontally by 360 degrees, while shifting the tray downward and its elbow upward. Then it continues the tray's horizontal rotation by additional 360 degrees in the same direction, while shifting the tray upward and its elbow downward to their original position. (a) Try to execute this maneuver with a loaded tray. (b) Identify the upper hemisphere of S^3 with D^3. Recognize that identifying antipodal points on the equator of the upper hemisphere is equivalent to the procedure for constructing $I\!RP^3$. But this is precisely the periodicity rule of the $\boldsymbol{\theta}$ parametrization. 2. When one uses quaternions, the parametrization of $SO(3)$ is in terms of S^3.

Exercise 4. verify that the rigid-body transformation is the general form of an orientation preserving isometric embedding of $I\!R^3$. [See Rees, *Notes on Geometry*]

Exercises 5. (a) Prove that the path connectivity property of a c-obstacle propagates. (b) Prove that convexity propagates based on the fact that $\mathsf{dst}(x, \mathcal{O})$ is a convex function when $\mathcal{O}$ is convex. Recall that a function, $f(x)$, is convex when its *epigraph* (the set of points on or above the graph of the function) is a convex set.

Exercise 6. Consider the construction of the c-obstacle boundary in the case when the boundary of the planar body $\mathcal{B}$ is a smooth convex curve, and $\mathcal{O}$ is a planar disc of radius r. Show that the curve traced by $\mathcal{B}$'s origin in $\mathcal{F}_W$ is: $d(s,\theta) = x_0 - R(\theta)\big(\beta(s) + rJ\beta'(s)\big)$, where $R(\theta) \in SO(2)$ $\mathcal{B}$'s orientation matrix.

Solution 6. Let θ_0 be a particular orientation of the planar body $\mathcal{B}$. When $\mathcal{B}$ traces $\mathcal{O}$'s perimeter (i.e., slides along the boundary of $\mathcal{O}$) at a fixed orientation θ_0, the curve traced by $\mathcal{B}$'s contact point in $\mathcal{F}_W$ is: $x(s) = R(\theta_0)b(s) + d(s)$ for $s \in I\!R$. The c-obstacle boundary is the curve traced by $\mathcal{B}$'s origin during this motion: $d(s) = x(s) - R(\theta_0)b(s)$. Since the contact normals of $\mathcal{O}$ and $\mathcal{B}$ are collinear at $x(s)$, $\mathcal{O}$'s center point satisfies the equation: $x_0 = x(s) + rR(\theta_0)J\beta'(s)$, since $J\beta'(s)$ points into $\mathcal{O}$. Substituting for $x(s)$ in the expression for $d(s)$ gives: $d(s) = x_0 - rR(\theta_0)J\beta'(s) - R(\theta_0)b(s) = x_0 - R(\theta_0)\big(b(s) + rJ\beta'(s)\big)$.

Exercise 7. (a). Verify that the Jacobian of $\varphi(s,\theta) = \big(d(s,\theta),\theta\big)$ has full rank. From this one can conclude that it locally parametrizes a smooth surface. (b) Generalize $\varphi(s,\theta) = \big(d(s,\theta),\theta\big)$ to convex and smooth 2D bodies. (c) Use the fact that Lipschitz continuous functions are piecewise smooth to conclude that the c-obstacles are bounded by piecewise smooth surfaces. (d) Show that when $\mathcal{B}$ and $\mathcal{O}$ are smooth convex shapes, the c-obstacle boundary is a single smooth surface.

Solution 7. We must verify that the tangent vectors $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial \theta}\varphi(s,\theta)$ are linearly independent and therefore span the tangent plane to $\mathcal{CO}$'s boundary at $\varphi(s,\theta)$.

$$\frac{\partial}{\partial s}\varphi(s,\theta) = \begin{pmatrix} -R(\theta)\big(\beta'(s) + rJ\beta''(s)\big) \\ 0 \end{pmatrix} = -(1 + r\kappa_{\mathcal{B}}(s)) \begin{pmatrix} R(\theta)\beta'(s) \\ 0 \end{pmatrix},$$

where we used the fact that $J\beta''(s)$ is collinear with $\beta'(s)$, and that $\kappa_{\mathcal{B}}(s) = \beta'(s) \cdot J\beta''(s)$ is the curvature of $\mathcal{B}$ at $\beta(s)$.

$$\frac{\partial}{\partial \theta}\varphi(s,\theta) = \begin{pmatrix} -JR(\theta)\big(\beta(s) + rJ\beta'(s)\big) \\ 1 \end{pmatrix},$$

where we used the identity $R'(\theta) = JR(\theta)$. Since $\kappa_{\mathcal{B}}(s) > 0$ for a convex $\mathcal{B}$, the tangent vectors $\frac{\partial}{\partial s}\varphi(s,\theta)$ and $\frac{\partial}{\partial \theta}\varphi(s,\theta)$ are linearly independent.

Exercise 8. Let $\mathcal{B}$ be a convex polygon and $\mathcal{O}$ a stationary disc. In this case $\mathcal{CO}$'s boundary consists of two-dimensional patches meeting along one-dimensional curves. Identify the possible types of these patches. Write the (s,θ) parametrization of the patch generated by an edge of $\mathcal{B}$.

Solution 8. There are two types of smooth patches. An edge-patch generated by an edge of $\mathcal{B}$ sliding on $\mathcal{O}$, and a vertex-patch generated by a vertex of $\mathcal{B}$ sliding on $\mathcal{O}$. Let $\mathcal{O}$ have a radius r and center x_0. Consider now an edge of $\mathcal{B}$ having endpoints b_1 and b_2 and length L. Let $v = (b_2 - b_1)/L$ be the edge's direction. Then $\beta(s) = b_1 + sv$ for $0 \leq s \leq L$ parametrizes the edge in $\mathcal{F}_B$, with $\beta'(s) = 1$. Since the edge can touch $\mathcal{O}$ from the outside at any orientation θ, the parameter θ varies freely in $I\!R$. Following the solution approach of the previous exercise, $d(s,\theta) = x_0 - R(\theta)(b_1 + sv + rJv)$ for $0 \leq s \leq L$ and $\theta \in I\!R$. Note that $d(s,\theta)$ is linear in s, implying that the patch $\varphi(s,\theta) = (d(s,\theta),\theta)$ is a ruled surface in this case.

Exercise 9. Compute the Jacobian, $DX_b(q) = \frac{dX_b q}{dq}$, in Equation (2.2 for the case of 3-dimensional objects. Obtain the equivalent formula for 2-dimensional objects as a special case of the 3D formula, by embedding the planar environment in a horizontal plane passing through the origin in three-dimensions.

Exercise 10. Consider the c-obstacle normal $\eta(q)$ in Equation (2.3) for the case in which the origin of the body fixed reference frame, $\mathcal{F}_B$, lies along the contact normal. Verify that the tangent plane to $\mathcal{CO}$ at q is vertical in this case, implying that an instantaneous rotation of $\mathcal{B}$ about $\mathcal{F}_B$'s origin is tangent to $\mathcal{CO}$'s boundary at q.

Appendix: Details of Proofs

This appendix contains a derivation of the c-obstacle curvature formula in the 2D case.

Lemma 2.3.1. *Let $q(t)$ be a c-space trajectory on $\mathcal{S}$, and let $x(t)$ be $\mathcal{B}$'s contact point with $\mathcal{O}$ along $q(t)$. Then the contact point velocity is given by*

$$\dot{x} = \tfrac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\left[I \;\; JR(\theta)b_c\right]\dot{q} \qquad \dot{q} = \tfrac{d}{dt}q(t),$$

where $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at $x(t)$, and b_c is $\mathcal{B}$'s center of curvature at $x(t)$ expressed in $\mathcal{F}_B$; I is a 2×2 identity matrix and $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$.

Proof: Let $q(t)$ be a curve on $\mathcal{S}$. As $\mathcal{B}$ moves along $q(t)$, the contact point satisfies the formula: $x(t) = X(q(t),d(t)) = R(\theta(t))b(t) + d(t)$. Taking the time derivative of both sides gives $\dot{x} = DX_b(q)\dot{q} + R(\theta)\dot{b}$. In order to obtain an expression for $\dot{x}$ as a function of $\dot{q}$, we need a second equation relating $(\dot{x}, \dot{b})$ to $\dot{q}$. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $q(t)$, the outward unit normal to $\mathcal{O}$'s boundary at x, $n(x)$, must match the *inward* unit normal to $\mathcal{B}$ at x. Let $\bar{n}(b)$ denote the outward unit normal to $\mathcal{B}$ at b, expressed in $\mathcal{F}_B$. Then $-R(\theta)\bar{n}(b)$ is the direction of $\mathcal{B}$'s *inward* unit normal at x (in world reference frame coordinates), and therefore $n(x(t)) = -R(\theta(t))\bar{n}(b(t))$ along $q(t)$. Taking the time derivative of both sides (and suppressing t) gives

$$\tfrac{d}{dt}n(x) = -\big(JR(\theta)\bar{n}(b)\dot{\theta} + R(\theta)\tfrac{d}{dt}\bar{n}(b)\big),$$

where we used the formula $\dot{R}(\theta) = JR(\theta)\dot{\theta}$. Based on the definition of curvature,

$$\kappa_{\mathcal{O}}\dot{x} = -\big(JR(\theta)\bar{n}(b)\dot{\theta} + \kappa_{\mathcal{B}}R(\theta)\dot{b}\big).$$

Substituting $R(\theta)\dot{b} = \dot{x} - DX_b(q)\dot{q}$ in the latter equation gives

$$(\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}})\dot{x} = \kappa_{\mathcal{B}}DX_b(q)\dot{q} - JR(\theta)\bar{n}(b)\dot{\theta}.$$

Substituting $DX_b(q)\dot{q} = \dot{d} + JRb\dot{\theta}$ and pulling $\kappa_{\mathcal{B}}$ as a common factor gives

$$(\kappa_{\mathcal{B}} + \kappa_{\mathcal{O}})\dot{x} = \kappa_{\mathcal{B}}\big\{\dot{d} - JR\big(b - r_{\mathcal{B}}\bar{n}(b)\big)\dot{\theta}\big\}.$$

The term $b - r_{\mathcal{B}}\bar{n}(b)$ is the position of $\mathcal{B}$'s center-of-curvature in $\mathcal{F}_B$. Substituting $b_c = b - r_{\mathcal{B}}\bar{n}(b)$ gives $\dot{x} = \kappa_{\mathcal{B}}/(\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}})\,[I \;\; JR(\theta)b_c]\,\dot{q}$, where $\dot{q} = (\dot{d}, \dot{\theta})$. $\square$

The next lemma gives the formula for the c-obstacle curvature in the 2D case.

Lemma 2.3.2. *Let $\mathcal{S}$ be the boundary of $\mathcal{CO}$, let $q \in \mathcal{S}$, and let $\eta(q)$ be the normal to $\mathcal{S}$ at q. The curvature form of $\mathcal{S}$ at q is given by*

$$\kappa(q,\dot{q}) = \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\,\dot{q}^T \begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}I & \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}JRb_c \\ \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb_c)^T & (\kappa_{\mathcal{O}}Rb - n(x))^T(\kappa_{\mathcal{B}}Rb + n(x)) \end{bmatrix} \dot{q} \qquad \dot{q} \in T_q\mathcal{S},$$

where $x = X(q,b)$ is $\mathcal{B}$'s contact point with $\mathcal{O}$, $\kappa_{\mathcal{B}}$ and $\kappa_{\mathcal{O}}$ are the curvatures of $\mathcal{B}$ and $\mathcal{O}$ at x, $n(x)$ is the contact normal at x, and b_c is $\mathcal{B}$'s center of curvature at x expressed in $\mathcal{F}_B$; I is a 2×2 identity matrix and $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Proof: Let $q(t)$ be a curve on $\mathcal{S}$ such that $q(0) = q$ and $\tfrac{d}{dt}\big|_{t=0} q(t) = \dot{q}$. Based on the definition of $\kappa(q,\dot{q})$, we have to compute the derivative

$$\frac{d}{dt}\bigg|_{t=0} \hat{\eta}(q(t)) = \frac{d}{dt}\bigg|_{t=0} \tfrac{1}{\|\eta(q(t))\|}\eta(q(t)) = \frac{1}{\|\eta(q)\|}\,\big[I - \hat{\eta}(q)\hat{\eta}(q)^T\big]\,\frac{d}{dt}\bigg|_{t=0} \eta(q(t)).$$

Since $\big[I - \hat{\eta}(q)\hat{\eta}(q)^T\big]\dot{q} = \dot{q}$ on $T_q\mathcal{S}$, the curvature form can be equivalently written as

$$\kappa(q,\dot{q}) = \frac{1}{\|\eta(q)\|}\,\dot{q}\cdot\frac{d}{dt}\bigg|_{t=0} \eta(q(t)) \qquad \dot{q} \in T_q\mathcal{S}.$$

Recall now that $\eta(q) = DX_b^T(q)n(x)$, where $DX_b^T(q) = [I\ JRb]^T$ and $n(x)$ is the unit normal to $\mathcal{O}$ at x. Thus we have to compute the derivative

$$\frac{d}{dt}\bigg|_{t=0} \eta(q(t)) = DX_b^T(q)\, \frac{d}{dt}\bigg|_{t=0} n(x(t)) + \left(\frac{d}{dt}\bigg|_{t=0} DX_b^T(q(t))\right) n(x).$$

Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $q(t)$, the contact $x(t)$ moves along $\mathcal{O}$'s boundary, and $\frac{d}{dt}\big|_{t=0} n(x(t)) = \kappa_{\mathcal{O}}\dot{x}$ in the first summand. Substituting $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I\ JRb_c]\dot{q}$ according to the contact velocity formula gives

$$DX_b^T(q)\,\frac{d}{dt}\bigg|_{t=0} n(x(t)) = \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} I \\ (JRb)^T \end{bmatrix}[I\ JRb_c]\,\dot{q} = \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} I & JRb_c \\ (JRb)^T & b\cdot b_c \end{bmatrix}\dot{q},$$

where we used the identities $R^TR = J^TJ = I$. In the second summand, $\frac{d}{dt}\big|_{t=0} DX_b^T(q) = \big[O\ J\frac{d}{dt}\big|_{t=0}(Rb)\big]^T$, where O is a 2×2 matrix of zeroes. Since $\mathcal{B}$ maintains continuous contact with $\mathcal{O}$ along $q(t)$, the contact point satisfies the equation: $x(t)=R(\theta(t))b(t)+d(t)$. Taking the time derivative of both sides gives $\dot{x} = \frac{d}{dt}(Rb) + \dot{d}$. Substituting $\dot{x} = \frac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I\ JRb_c]\dot{q}$ according to the contact velocity formula gives

$$\frac{d}{dt}(Rb) = \tfrac{\kappa_{\mathcal{B}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[I\ JRb_c]\,\dot{q} - \dot{d} = \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}[-\kappa_{\mathcal{O}}I\ \ \kappa_{\mathcal{B}}JRb_c]\,\dot{q}.$$

The second summand is thus

$$\big(\tfrac{d}{dt}\big|_{t=0} DX_b^T(q(t))\big)n(x) = \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} O \\ n^T(x)J\,[-\kappa_{\mathcal{O}}I\ \ \kappa_{\mathcal{B}}JRb_c]\,\dot{q} \end{bmatrix} = \tfrac{-1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} O & \vec{0} \\ \kappa_{\mathcal{O}}n^T(x)J & \kappa_{\mathcal{B}}n^T(x)Rb_c \end{bmatrix}\dot{q},$$

where we used the identity $J^2 = -I$. Substituting for the two summands in the derivative $\frac{d}{dt}\big|_{t=0}\eta(q(t))$ gives

$$\begin{aligned}\tfrac{d}{dt}\big|_{t=0}\eta(q(t) &= \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} O & \vec{0} \\ -\kappa_{\mathcal{O}}n^T(x)J & -\kappa_{\mathcal{B}}n^T(x)Rb_c \end{bmatrix}\dot{q} + \tfrac{\kappa_{\mathcal{B}}\kappa_{\mathcal{O}}}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} I & JRb_c \\ (JRb)^T & b\cdot b_c \end{bmatrix}\dot{q} \\ &= \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}I & \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}JRb_c \\ -\kappa_{\mathcal{O}}n^T(x)J + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb)^T & -\kappa_{\mathcal{B}}n^T(x)Rb_c + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}b\cdot b_c \end{bmatrix}\dot{q}.\end{aligned}$$

The expression on the lower left simplifies as follows. Let $\bar{n}(b)$ denote the outward unit normal to $\mathcal{B}$'s boundary at b, expressed in $\mathcal{F}_B$. Then $n(x) = -R(\theta)\bar{n}(b)$, and we have

$$-\kappa_{\mathcal{O}}n^T(x)J + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb)^T = \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}\big(-r_{\mathcal{B}}\bar{n}(b) + b\big)^TR^TJ^T = \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb_c)^T,$$

where we substituted $b_c = b - r_{\mathcal{B}}\bar{n}(b)$ for $\mathcal{B}$'s center of curvature at x. The expression on the lower right simplifies as follows

$$-\kappa_{\mathcal{B}}n^T(x)Rb_c + \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}b\cdot b_c = (\kappa_{\mathcal{O}}Rb - n(x))^T(\kappa_{\mathcal{B}}Rb_c) = (\kappa_{\mathcal{O}}Rb - n(x))^T(\kappa_{\mathcal{B}}Rb + n(x)),$$

where we substituted $\kappa_{\mathcal{B}}Rb_c = \kappa_{\mathcal{B}}R(b - r_{\mathcal{B}}\bar{n}(b)) = \kappa_{\mathcal{B}}Rb + n(x)$. Substituting the simplified terms in the derivative $\frac{d}{dt}\big|_{t=0}\eta(q(t))$ gives

$$\frac{d}{dt}\bigg|_{t=0}\eta(q(t) = \tfrac{1}{\kappa_{\mathcal{B}}+\kappa_{\mathcal{O}}}\begin{bmatrix} \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}I & \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}JRb_c \\ \kappa_{\mathcal{B}}\kappa_{\mathcal{O}}(JRb_c)^T & (\kappa_{\mathcal{O}}Rb-n(x))^T(\kappa_{\mathcal{B}}Rb+n(x)) \end{bmatrix}\dot{q}.$$

Finally, pre-multiplying by $1/\|\eta(q)\|$ and $\dot{q}$ gives the c-obstacle curvature form. $\square$

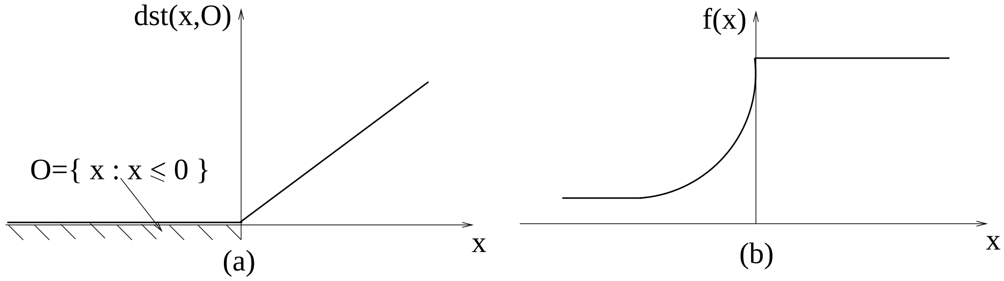

Figure 7: (a) The minimal distance function $\mathsf{dst}(x, \mathcal{O})$ is piecewise smooth and Lipschitz continuous. (b) A piecewise smooth function which is non-Lipschitz at $x = 0$.

Appendix: An Introduction to Non-Smooth Analysis

This appendix introduces some useful non-smooth analysis tools concerning Lipschitz continuous functions.

Definition 3 (Lipschitz Function)**.** *A continuous function $f(x) : \mathbb{R}^m \to \mathbb{R}$ is* Lipschitz continuous *at x if there exists a constant $k > 0$ such that $|f(x_1) - f(x_2)| \le k\|x_1 - x_2\|$ for all x_1 and x_2 in a neighborhood of x. A continuous function $f(x) : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous at x if $\|f(x_1) - f(x_2)\| \le k\|x_1 - x_2\|$ in a neighborhood of x.*

In both cases k is a *Lipschitz constant* for f at x. According to Rademacher's Theorem from functional analysis, all Lipschitz continuous functions are piecewise differentiable. Lipschitz continuous functions are essentially piecewise smooth functions whose slope is bounded away from infinity (see Figure 7).

Example: Let us verify that $f(x) = |x|$ is Lipschitz continuous. Using the identity $|x| = \sqrt{x^2}$, one can write $|f(x_1) - f(x_2)| = \sqrt{(|x_1| - |x_2|)^2} = \sqrt{|x_1|^2 - 2|x_1||x_2| + |x_2|^2} \le \sqrt{x_1^2 - 2x_1x_2 + x_2^2} = |x_1 - x_2|$. Thus $f(x) = |x|$ is Lipschitz continuous with $k = 1$.

Some useful properties of Lipschitz continuous are as follows.

1. A differentiable function $f(x)$ is automatically Lipschitz continuous, with $k = \|\nabla f(x)\|$. Conversely, if $f(x)$ is Lipschitz continuous and $\partial f(x)$ contains a single vector, then f is differentiable at x and $\partial f(x) = \nabla f(x)$.

2. The composition of two Lipschitz continuous functions is Lipschitz continuous. To see this fact, let $g(y) : \mathbb{R}^n \to \mathbb{R}$ and $h(x) : \mathbb{R}^m \to \mathbb{R}^n$ be Lipschitz continuous functions with Lipschitz constants k_g and k_h, and let $f(x) = g(h(x))$. Then
$$|f(x_1) - f(x_2)| = |g(h(x_1)) - g(h(x_2))| \le k_g\|h(x_1) - h(x_2)\| \le k_g k_h\|x_1 - x_2\|,$$
thus proving that $f(x) = g(h(x))$ is Lipschitz continuous.

3. Let $F(x) = \min_{t \in \mathcal{T}}\{f_t(x)\}$ be the pointwise minimum over a parametrized family of Lipschitz continuous functions, where $\mathcal{T}$ is a discrete or a continuous set. Then $F(x)$

is Lipschitz continuous. To see this fact, let k_t be the Lipschitz constant of $f_t(x)$ for $t \in \mathcal{T}$. Let $F(x_1) = f_{t_1}(x)$ and let $F(x_2) = f_{t_2}(x)$ for some $t_1, t_2 \in \mathcal{T}$. We have to show that $|F(x_1) - F(x_2)| = |f_{t_1}(x_1) - f_{t_2}(x_2)| \leq k\|x_1 - x_2\|$ for some $k > 0$. There are two cases to consider. If $f_{t_1}(x_1) - f_{t_2}(x_2) \geq 0$, then $|f_{t_1}(x_1) - f_{t_2}(x_2)| \leq |f_{t_2}(x_1) - f_{t_2}(x_2)| \leq k_{t_2}\|x_1 - x_2\|$ (since $f_{t_1}(x_1) \leq f_{t_2}(x_1)$). If $f_{t_1}(x_1) - f_{t_2}(x_2) < 0$, then $|f_{t_2}(x_2) - f_{t_1}(x_1)| \leq |f_{t_1}(x_2) - f_{t_1}(x_1)| \leq k_{t_1}\|x_1 - x_2\|$ (since $f_{t_2}(x_2) \leq f_{t_1}(x_2)$). It follows that $F(x)$ is Lipschitz continuous with $k = \max\{k_{t_1}, k_{t_2}\}$.

Next we define the generalized gradient of a Lipschitz continuous function.

Definition 4 (Generalized Gradient)**.** *Let $f(x) : I\!R^m \to I\!R$ be a Lipschitz continuous function. Let $x \in I\!R^m$ be surrounded by open sets $\mathcal{S}_1 \ldots \mathcal{S}_N$ containing x on their common boundary, such that f is differentiable on each of these sets. Then the generalized gradient of f at x, denoted $\partial f(x)$, is given by*

$$\partial f(x) = \sum_{i=1}^{N} \lambda_i \lim_{x_j \to x, x_j \in S_i} \nabla f(x_j) \quad 0 \leq \lambda_i \leq 1 \text{ for } i = 1 \ldots N \text{ and } \textstyle\sum_{i=1}^{N} \lambda_i = 1.$$

(i.e. $\partial f(x)$ is the convex combination of the limits $\lim_{x_j \to x} \nabla f(x_j)$, each taken along a sequence $\{x_j\}$ approaching x inside the set $\mathcal{S}_i$ for $i = 1 \ldots N$).

Example: Let us compute the generalized gradient of $f(x) = |x|$. Clearly $f'(x) = -1$ for $x < 0$ and $f'(x) = 1$ for $x > 0$. Based on the definition, $\partial f(0)$ is the convex combination of -1 and 1, giving the interval $[-1, 1]$. We can thus write

$$\partial f(x) = \begin{cases} -1 & x < 0 \\ [-1, 1] & x = 0 \\ +1 & x > 0 \end{cases} \qquad \text{for } f(x) = |x|.$$

Let $\mathsf{bdy}(\mathcal{O})$ denote the boundary of a fixed set $\mathcal{O}$. The following lemma specifies the generalized gradient of the minimal distance function, $\mathsf{dst}(x, \mathcal{O})$, at a point $x \in \mathsf{bdy}(\mathcal{O})$.

Lemma .0.3. *Let $\mathcal{O}$ be a closed set having a non-empty interior and a smooth boundary. The function $\mathsf{dst}(x, \mathcal{O}) = \min_{y \in \mathcal{O}}\{\|x - y\|\}$ is Lipschitz continuous in x, and at $x \in \mathsf{bdy}(\mathcal{O})$ its generalized gradient is given by*

$$\partial \mathsf{dst}(x, \mathcal{O}) = s \cdot n(x) \quad \text{for } 0 \leq s \leq 1,$$

where $n(x)$ is the outward unit normal to the boundary of $\mathcal{O}$ at x.

Proof: We first verify that $\mathsf{dst}(x, \mathcal{O})$ is Lipschitz continuous in x. Let x_1 and x_2 be any two points in a neighborhood of x. We have to show that $|\mathsf{dst}(x_1, \mathcal{O}) - \mathsf{dst}(x_2, \mathcal{O})| \leq k\|x_1 - x_2\|$ for some $k > 0$. Let $x_i^* \in \mathcal{O}$ minimize the distance of x_i from $\mathcal{O}$, so that $\mathsf{dst}(x_i, \mathcal{O}) = \|x_i - x_i^*\|$ for $i = 1, 2$. Since $\mathsf{dst}(x_1, \mathcal{O})$ is the minimal distance of x_1 from $\mathcal{O}$ and $x_2^* \in \mathcal{O}$, $\mathsf{dst}(x_1, \mathcal{O}) \leq \|x_1 - x_2^*\|$. But $\|x_1 - x_2^*\| = \|(x_1 - x_2) - (x_2^* - x_2)\| \leq \|x_1 - x_2\| +$

$\|x_2^* - x_2\|$. Since $\mathsf{dst}(x_2, \mathcal{O}) = \|x_2 - x_2^*\|$, we obtain that $\mathsf{dst}(x_1, \mathcal{O}) \leq \|x_1 - x_2\| + \mathsf{dst}(x_2, \mathcal{O})$. Equivalently, $\mathsf{dst}(x_1, \mathcal{O}) - \mathsf{dst}(x_2, \mathcal{O}) \leq \|x_1 - x_2\|$. Using the same argument with the roles of x_1 and x_2 exchanged gives $\mathsf{dst}(x_2, \mathcal{O}) - \mathsf{dst}(x_1, \mathcal{O}) \leq \|x_1 - x_2\|$. Hence $|\mathsf{dst}(x_1, \mathcal{O}) - \mathsf{dst}(x_2, \mathcal{O})| \leq \|x_1 - x_2\|$, implying that $\mathsf{dst}(x, \mathcal{O})$ is Lipschitz continuous with $k = 1$.

The generalized gradient of $\mathsf{dst}(x, \mathcal{O})$ at $x \in \mathsf{bdy}(\mathcal{O})$ is the convex combination of the limits $\lim_{x_j \to x} \nabla \mathsf{dst}(x_j, \mathcal{O})$, each taken along a sequence $\{x_j\}$ approaching x through differentiable points of $\mathsf{dst}(x, \mathcal{O})$. Since $\mathsf{dst}(x, \mathcal{O})$ is identically zero in the interior of $\mathcal{O}$, $\lim_{x_j \to x} \nabla \mathsf{dst}(x_j, \mathcal{O}) = \vec{0}$ along along any sequence $\{x_j\}$ approaching x from the interior of $\mathcal{O}$. Next consider a sequence $\{x_j\}$ approaching x from the exterior of $\mathcal{O}$. Since the boundary of $\mathcal{O}$ is smooth, it has a unit outward normal, denoted $n(y)$, which varies smoothly with $y \in \mathsf{bdy}(\mathcal{O})$. The collection of segments $y + sn(y)$ such that $0 \leq s \leq \delta$ for some small $\delta > 0$ forms an exterior neighborhood of $\mathcal{O}$, denoted $\mathcal{N}$. Any point in $\mathcal{N}$ has a unique closest point on $\mathsf{bdy}(\mathcal{O})$. When a sequence $\{x_j\}$ approaches x from the exterior of $\mathcal{O}$, it must eventually lie within $\mathcal{N}$. At each point $x_j \in \mathcal{N}$, the minimal distance $\mathsf{dst}(x_j, \mathcal{O}) = \min_{y \in \mathcal{O}}\{\|x_j - y\|\}$ is attained at a unique point $y_j \in \mathsf{bdy}(\mathcal{O})$. Since $\mathsf{dst}(x_j, \mathcal{O})$ is a minimum over a parametrized family of functions, $\partial \mathsf{dst}(x_j, \mathcal{O}) = \nabla \|x - y_j\|$ evaluated at $x = x_j$. Thus $\partial \mathsf{dst}(x_j, \mathcal{O}) = (x_j - y_j)/\|x_j - y_j\|$, which is a unit vector pointing from y_j toward x_j. Since $\partial \mathsf{dst}(x_j, \mathcal{O})$ contains a single vector, $\mathsf{dst}(x, \mathcal{O})$ is differentiable at $x = x_j$, and $\partial \mathsf{dst}(x_j, \mathcal{O}) = \nabla \mathsf{dst}(x_j, \mathcal{O})$. In particular, $\lim_{x_j \to x} \nabla \mathsf{dst}(x_j, \mathcal{O}) = \lim_{x_j \to x} n(x_j) = n(x)$, where $n(x)$ is the unit outward normal to $\mathcal{O}$ at x. Hence $\partial \mathsf{dst}(x, \mathcal{O})$ is the convex combination of 0 and $n(x)$, which gives the line segment $s \cdot n(x)$ for $0 \leq s \leq 1$. $\square$

Bibliographical Notes

The generalized gradient has its origin with the notion of *subgradient* of convex analysis [10]. The generalized gradient is discussed in Clarke's *Optimization and Nonsmooth Analysis* [3], which also provides various tools for its computation.

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