Chapter 6

Introduction to Secure Grasps

Having developed in detail the theory of *equilibrium grasps,* our next step is to develop a theory of *secure grasps.* Secure grasps should be able to keep the object safely contained within the grasping fingers in the face of all possible perturbations that might arise while the object is transported and manipulated. For instance, a multi-finger robot hand is required to safely transport objects in the presence of disturbances generated by movement of the robot arm on which the grasping mechanism is mounted. Similarly, a fixturing system must hold workpieces within a prescribed position tolerance in the face of disturbances caused by manufacturing or assembly operations. During legged locomotion, the moving robot is required to safely support itself in the presence of disturbances generated by movement of its free limbs.

This chapter introduces two complementary types of secure grasps: *immobilizing* and *wrench resistant* grasps[1]. In *immobilizing grasps*, which are introduced in Section 6.1, the fingers are arranged so as to prevent any movement of the object with respect to the grasping fingers. In the complementary *wrench resistant grasps,* introduced in Section 6.2, the fingers are arranged so that they can potentially counterbalance all possible forces and torques that might disturb the object within the grasp, thereby *resisting* the tendency of the disturbing forces to wrench the object fromf within the grasp. In order to focus on the essential ideas, most of this chapter, as well as Part II of the book, consider frictionless contacts. While no contacts are truly frictionless, a secure frictionless grasp provides a conservative approximation in cases where the magnitude of the contact friction is small, or the grasp is lightly loaded. Moreover, frictionless grasps can be analyzed using the geometric c-space techniques introduced in Part I of the book. Their study thus serves as a useful introduction to the key grasp safety issues, which are extended to frictional grasps in Part III of the book. Section 6.3 establishes that in the case of frictionless contacts, immobilizing and resistant grasps are perfectly *dual* notions (this duality breaks down in the presence of friction).

[1]In the grasping and fixturing literature, the term *form closure* is also used for first-order immobilization. Similarly, the term *force closure* has been traditionally used for our notion of wrench resistant grasps. See Section 6.5 for additional discussion of this terminology

6.1 Immobilizing Grasps

The *mobility* of a mechanism composed of moving mechanical parts is the number of independent degrees-of-freedom (DOF) needed to uniquely describe the internal motions of the mechanism's components. When a mechanism is *immobile*, it forms a rigid structure where no internal movement of its constituent parts is possible. In grasping mechanics, we seek to analyze the mobility of a grasped object, $\mathcal{B}$, relative to stationary finger bodies, $\mathcal{O}_1, \ldots, \mathcal{O}_k$, under the frictionless rigid body model. This type of mobility is most naturally analyzed in the object's configuration space (c-space), which uses studied in earlier chapters. The c-space view of mobility starts with the following definition of the set of free configurations.

Definition 1 (**Free C-Space**). *Let a rigid object $\mathcal{B}$ be contacted by rigid and stationary fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. Let $\mathcal{CO}_1, \ldots, \mathcal{CO}_k$ be the finger c-obstacles in $\mathcal{B}$'s c-space. The* **free c-space***, $\mathcal{F}$, is the set of configurations*

$$\mathcal{F} = \mathcal{C} - \bigcup_{i=1}^{k} \mathsf{int}(\mathcal{CO}_i) \qquad m = 3 \text{ or } 6,$$

where $\mathsf{int}(\mathcal{CO}_i)$ denotes the interior of $\mathcal{CO}_i$.

The interior of the free c-space is an open subset of $\mathcal{C}$ which consists of $\mathcal{B}$'s contact-u100free configurations. The boundary of the free c-space is the union of the finger c-obstacle boundaries, $\mathsf{bdy}(\mathcal{F}) = \cup_{i=1}^{k} \mathsf{bdy}(\mathcal{CO}_i)$. Any c-space path which lies on $\mathsf{bdy}(\mathcal{CO}_i)$ represents a perfectly feasible motion of $\mathcal{B}$ which maintains surface contact with the stationary finger body $\mathcal{O}_i$. When $\mathcal{B}$ is contacted by k fingers at a configuration q_0, the point q_0 lies at the intersection of the finger c-obstacle boundaries, $q_0 \in \cap_{i=1}^{k} \mathsf{bdy}(\mathcal{CO}_i)$. The free motions available to $\mathcal{B}$ are defined as follows.

Definition 2 (**Free Motions**). *Let $\mathcal{B}$ be located at a configuration q_0 in contact with stationary fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. Let $\mathcal{D}$ be a small m-dimensional ball in $\mathcal{B}$'s c-space centered at q_0. The* **free c-space motions** *of $\mathcal{B}$ at q_0 are those local c-space paths which start at q_0 and lie in $\mathcal{D} \cap \mathcal{F}$.*

The free motions are those local motions along which $\mathcal{B}$ either breaks away or maintains surface contact with the stationary fingers (some examples are discussed below). When $\mathcal{B}$ has available free motions at a given grasp, there exist perturbing forces that can induce these motions and thus allow $\mathcal{B}$ to escape the grasp. Conversely, grasp safety is ensured when the grasping fingers allow no free motions of $\mathcal{B}$. This is the notion of immobilizing grasps introduced in the following definition.

Definition 3 (**Immobilization**). *A rigid object $\mathcal{B}$ held at a configuration q_0 by rigid and stationary fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$ is* **immobilized** *when it has no free c-space motions at q_0.*

Equivalently, an object is immobilized when the finger c-obstacles completely surround the object's configuration q_0. The finger c-obstacles can isolate the point q_0 only when the physical fingers form a frictionless equilibrium grasp. This important property is stated in the following theorem (see appendix for a proof).

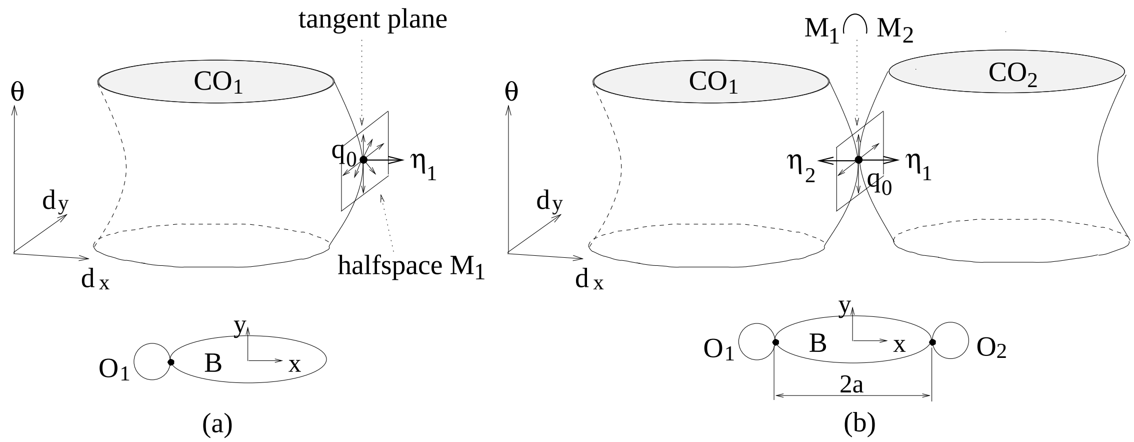

Figure 6.1: (a) The halfspace M_1 approximating the exterior of $\mathcal{CO}_1$ at q_0. (b) The 1'st-order approximation to the free motions, $M_1 \cap M_2$, at a two-finger equilibrium grasp.

Theorem 1 (**Equilibrium Grasp Immobilization**). *A necessary condition for a rigid object $\mathcal{B}$ to be immobilized by rigid fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$ is that the fingers hold the object in a feasible* **frictionless equilibrium grasp.**

To intuitively understand the theorem, consider the finger c-obstacles associated with a planar grasp. The outward normal to the i^{th} finger c-obstacle at q_0, $\eta_i(q_0)$, determines a tangent plane to $\mathcal{CO}_i$ at q_0. Let M_i denote the halfspace of tangent vectors based at q_0, which is bounded by the plane tangent to $\mathcal{CO}_i$ at q_0 and pointing away from $\mathcal{CO}_i$:

$$M_i = \{\dot{q} \in T_{q_0}\mathcal{C} : \eta_i(q_0) \cdot \dot{q} \geq 0\} \tag{6.1}$$

The halfspace M_i represents the first-order approximation to the free motions allowed by the i^{th} finger (Figure 6.1(a)). Let $M_{1,\ldots,k}$ denote the intersection of the free halfspaces associated with the k fingers:

$$M_{1,\ldots,k} \;=\; \cap_{i=1}^{k} M_i = \{\dot{q} \in T_{q_0}\mathcal{C} \;:\; \eta_i(q_0) \cdot \dot{q} \geq 0 \quad \text{for } i = 1 \ldots k\} \tag{6.2}$$

The set $M_{1\ldots k}$ forms a cone based at the origin of the object's tangent space, $T_{q_0}\mathcal{C}$. When $M_{1,\ldots,k}$ has a non-empty interior, any $\dot{q} \in T_{q_0}\mathcal{C}$ pointing into this interior determines a local motion which causes $\mathcal{B}$ to break contact with all k fingers. For $M_{i,\ldots,k}$ to have an empty interior, the c-obstacle normals must be *linearly dependent* at q_0. But $\eta_1(q_0), \ldots, \eta_k(q_0)$ are collinear with the wrenches generated by finger forces acting along $\mathcal{B}$'s inward contact normals. Hence $\mathcal{CO}_1, \ldots, \mathcal{CO}_k$ can isolate the point q_0 only when $\mathcal{B}$ is held in a frictionless equilibrium grasp. The theorem is illustrated with the following examples.

Example—a non-immobilizing equilibrium grasp: Consider the frictionless two-finger equilibrium grasp of an ellipse shown in Figure 6.1(b). The figure also shows the c-space geometry of this grasp, with q_0 being the object's equilibrium grasp configuration. Consider the first-order approximation to the finger c-obstacles at q_0. Set the world and object frames at the ellipse center, with the x-axis aligned with the ellipse's major axis. The contact locations are $x_1 = (-a, 0)$ and $x_2 = (a, 0)$, where $2a$ is the ellipse major axis length. The

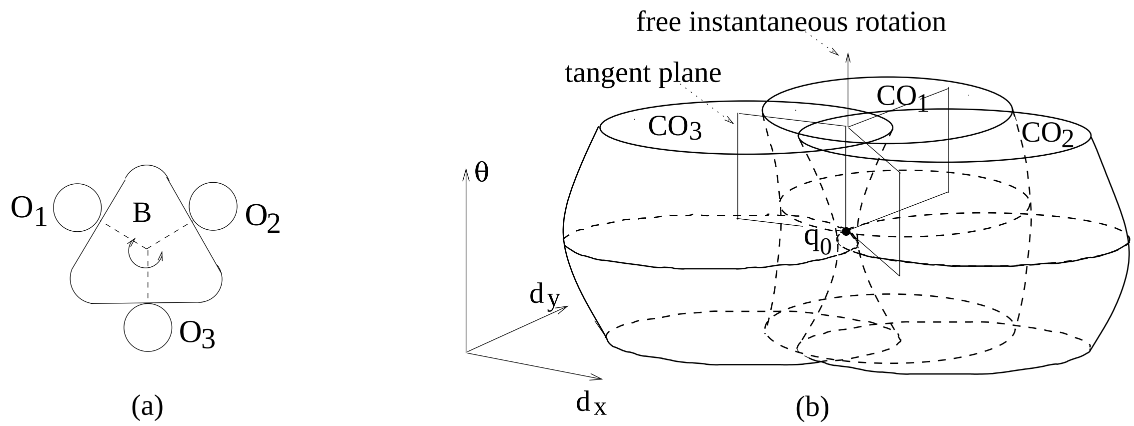

Figure 6.2: (a) A 3-finger equilibrium grasp of a triangular object. (b) The finger c-obstacles completely surround the object's configuration point q_0.

ellipse's inward unit normals at x_1 and x_2 are $n_1 = (1, 0)$ and $n_2 = (-1, 0)$. The finger c-obstacles outward normals are given by

$$\eta_1(q_0) = \begin{pmatrix} n_1 \\ x_1 \times n_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \eta_2(q_0) = \begin{pmatrix} n_2 \\ x_2 \times n_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix},$$

where $u \times v = u^T J v$ for $u, v \in \mathbb{R}^2$. The halfspace of tangent vectors based at q_0 and pointing away from $\mathcal{CO}_i$, $M_i = \{\dot{q} \in T_{q_0}\mathcal{C} : \eta_i(q_0) \cdot \dot{q} \geq 0\}$, is the first-order approximation to the free motions allowed by the i^{th} finger. When the fingers do *not* form an equilibrium grasp, $M_1 \cap M_2$ has a non-empty interior. When the fingers form an equilibrium grasp, the tangent vectors that lie in both halfspaces are given by

$$M_1 \cap M_2 = \left\{ \dot{q} \in T_{q_0}\mathbb{R}^m \ : \ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} v \\ \omega \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} v \\ \omega \end{pmatrix} \right\},$$

where $\dot{q} = (v, \omega)$ represents the linear and angular velocity of $\mathcal{B}$. Since the c-obstacle normals are antiparallel at the equilibrium grasp, $M_1 \cap M_2$ consists only of the tangent plane common to $\mathcal{CO}_1$ and $\mathcal{CO}_2$ at q_0, which has an empty interior.·

Frictionless equilibrium grasps are necessary but not sufficient for achieving object immobilization. For instance, the two-finger equilibrium grasp of Figure 6.1(b) is *non*-immobilizing, since the ellipse is free to escape along any combination of vertical translation and rotation about its center. This observation is supported by the first-order analysis, since any $\dot{q} \in M_1 \cap M_2$ consists of instantaneous vertical translation combined with instantaneous rotation about the contacts' midpoint. The next example describes an immobilizing equilibrium grasp.

Example—an immobilizing equilibrium grasp: Figure 6.2(a) shows a triangular object held in a frictionless equilibrium grasp by three disc fingers. The same first-order analysis implies that the triangular object is free to instantaneously rotate about the point of intersection of the finger contact normals. But clearly, the object is fully immobilized by the three fingers. As seen in the c-space geometry of Figure 6.2(b), the point q_0 is completely surrounded by the finger c-obstacles. How can this object be immobilized when the first-order analysis suggests that some instantaneous free motions are possible? The paradox can be resolved by realizing that the local free motions have *first order* (velocities, or tangent vectors) and *second order* (accelerations, or path curvature) properties. A proper analysis of mobility must include *both* properties of the local free motions. When curvature effects are taken into account, the triangular object is fully immobilized by the three fingers.·

Immobilization focuses on the rigid-body constraints imposed on the object free motions by the finger bodies. It is thus suited for judging grasp safety in *frictionless* situations, or in situations where friction effects cannot be relied upon (e.g. during high precision machining of fixtured parts). When friction effects are included, a grasp which does not satisfy the immobility condition can still form a secure grasp. This complementary notion of grasp security is next discussed.

6.2 Wrench Resistant Grasps

Rather than focus on the grasped object's free motions, one can alternatively analyze the fingers ability to resist perturbing forces and torques that might push or pull the object away from the grasping fingers. When every possible disturbance wrench applied to $\mathcal{B}$ can be counterbalanced by permissible finger forces, the grasped object can be safely kept within the grasping mechanism. The following definition of *wrench resistant grasps* is quite general and applies to both frictional and frictionless grasps.

Definition 4 (**Wrench Resistant Grasp**). *Let a rigid object $\mathcal{B}$ be held at a configuration q_0 by rigid fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The grasp is* **wrench resistant** *when any possible external wrench, $\boldsymbol{w}_{ext} \in T^*_{q_0}\mathcal{C}$, applied to $\mathcal{B}$ can be balanced by a set of feasible finger forces:*

$$G\vec{f} + \boldsymbol{w}_{ext} = \vec{0} \qquad \vec{f} = (f_1, \ldots, f_k) \in C_1 \times \cdots \times C_k, \tag{6.3}$$

where $T^*_{q_0}\mathcal{C}$ is the object's wrench space ($m=3$ or 6), G is the grasp map of the k-finger grasp, and $C_1, \ldots, C_k$ are the friction cones at the contacts, which define the set of feasible finger contact forces.

For the most common case of hard point contact model, the wrench resistance criterion is:

$$\forall \boldsymbol{w}_{ext} \in T^*_{q_0}\mathcal{C} \ \exists \vec{f} \in C_1 \times \cdots \times C_k \qquad \boldsymbol{w}_1 + \cdots + \boldsymbol{w}_k + \boldsymbol{w}_{ext} = \vec{0}, \tag{6.4}$$

where $\boldsymbol{w}_i = (f_i, x_i \times f_i)$ for $i = 1 \ldots k$ are the finger wrenches.

A grasp arrangement is wrench resistant when the grasp map, $G : C_1 \times \cdots \times C_k \to T^*_{q_0}\mathcal{C}$, maps the set of allowed finger forces *onto* the object's wrench space. In contrast with immobilizing grasps, wrench resistant grasps provide only a necessary condition for achieving secure grasps. Depending on the application, one must ensure that the grasping system actually generates the required reaction forces (this topic is further discussed below). Like immobilizing grasps, resistant grasps can only be achieved with equilibrium grasps—a frictionless equilibrium in the case of frictionless contacts, and a frictional equilibrium in the case of frictional contacts. This property is based on the following useful fact (see appendix for a proof).

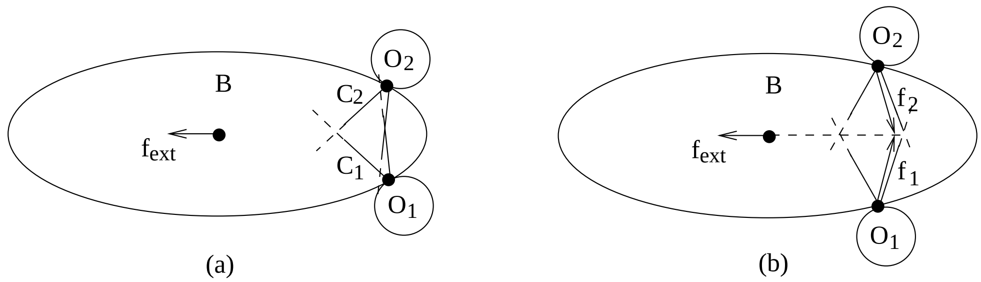

Figure 6.3: A frictional two-finger equilibrium which is (a) a non-resistant grasp, and (b) a wrench resistant grasp.

Proposition 6.2.1. *Let a rigid object $\mathcal{B}$ be held by rigid fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The fingers form a feasible* **equilibrium grasp** *iff their net wrench cone, $\mathcal{W} = \mathcal{W}_1 + \cdots + \mathcal{W}_k$,*[2] *contains a full subspace passing through the object's wrench space origin.*

Since the wrench cones associated with wrench resistant grasps span the entire wrench space, they must form equilibrium grasps as stated in the following corollary.

Corollary 6.2.2 (Wrench Resistant Equilibrium Grasp). *A necessary condition for a* **wrench resistant grasp** *is that the fingers hold the object in a feasible* **equilibrium grasp***—a frictionless equilibrium in the case of frictionless contacts, a frictional equilibrium in the case of frictional contacts.*

The following example illustrates the necessity of establishing an equilibrium grasp in order to achieve grasp resistance.

Example—necessity of equilibrium grasp: First consider the two-finger grasp of an ellipse shown in Figure 6.3(a). The indicated friction cones do not support opposing forces at the contacts, hence this is *not* a feasible equilibrium grasp. When an external force, f_{ext}, attempts to pull the ellipse away from the two fingers, there are no feasible finger forces, $f_1 \in C_1$ and $f_2 \in C_2$, that can possibly counterbalance this force. Next consider the two-finger grasp of the ellipse shown in Figure 6.3(b). The friction cones now support a two-finger equilibrium grasp. As discussed later in the book, in the case of frictional contacts a grasp is wrench resistant if and only if the contacts support an equilibrium whose forces lie in the *interior* of the respective friction cones. Since the opposing forces lie in the interior of their friction cones, this is a wrench resistant grasp. As indicated in Figure, 6.3 f_{ext} can be resisted by finger forces f_1 and f_2 which lie in their respective friction cones.·

Practical Implications of the Resistant Grasp Definition:

The definition of wrench resistant grasps makes rather strong assumptions about the design and operation of the grasping system. The grasping system must be able to quickly sense the perturbing wrench, rapidly compute a new set of finger forces that will cancel the perturbation, then react with the fingers so as to generate the required forces. Furthermore, the grasping system sensing and reaction cycle must keep pace with the natural time variations

[2]Each $\mathcal{W}_i$ is the i^{th} finger wrench cone, given by $\mathcal{W}_i = \{(f_i, x_i \times f_i) : f_i \in C_i\}$ for $i = 1 \ldots k$.

of the external perturbations. In light of this observation, the grasps of Definition 4 might aptly be called *actively wrench resistant grasps.*

In some grasping mechanisms, and in most fixtures, some or all of the contact forces are not actively generated and controlled by articulated fingers. Instead, the contact forces are generated by *passive* mechanical means. Such effects generally arise from two processes:

- A *preloading* process, where the object $\mathcal{B}$ is initially pressed against stationary "finger" bodies by an external agent (e.g. a factory worker squeezes a part against fixturing elements), ending at an equilibrium grasp. The preloading process establishes nonzero contact forces which lie within the allowed friction cones. When the fingers are stationary rigid bodies (e.g. fixturing elements), any subsequent external wrench perturbation acting on $\mathcal{B}$ (e.g. a factory worker attempting a machining operation) induces a perturbation of the contact forces. As long as the perturbed contact forces remain within their respective friction cones, the equilibrium grasp can be maintained.

- A *compliant reaction* process, where the finger forces vary in response to movements of the object $\mathcal{B}$. When $\mathcal{B}$ moves in response to perturbing wrenches, its contacts with the finger bodies are deformed. Reaction forces arise from a *stiffness relationship* that describes how the contact deformations generate reaction forces. Compliant finger mechanisms, or feedback algorithms that induce a compliant relationship at the joints of the finger mechanisms, can also contribute to system compliance.

A remark on compliant grasps: While this book considers the theory of grasping under the ideal rigid body model, let us sketch how compliance is included in grasp mechanics (see bibliographical notes). For small displacements of $\mathcal{B}$, the reaction force at the i^{th} contact can be modeled as:

$$f_i = -K_i(q - q_0) + f_i^0 \tag{6.5}$$

where f_i^0 denotes the preload force at the i^{th} contact, and K_i is the *stiffness matrix* associated with the i^{th} contact. Equation 6.5 can be seen as a model for a linear spring with spring constant K_i, and initial spring compression f_i^0. When $\mathcal{B}$ is held in a compliant equilibrium grasp at a configuration q_0, external wrench perturbations can only be balanced at equilibrium points which lie in the vicinity of q_0. Therefore, we are lead to the following more lenient notion of grasp resistance.

Definition 5 (**Locally Wrench Resistant Grasps).** *Let an object $\mathcal{B}$ be held by fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$ at an equilibrium grasp configuration q_0. Let $\mathcal{D}$ be a small m-dimensional ball in $\mathcal{B}$'s c-space centered at q_0. The grasp is* **locally wrench resistant** *if the contact reaction forces can counterbalance any external wrench in a bounded m-dimensional neighborhood centered at the object's wrench space origin at an equilibrium configuration which lies in $\mathcal{D}$.*

Locally wrench resistant grasps are required to resist only a bounded set of external wrenches, and may do so at an equilibrium which lies within a small neighborhood of q_0. This type of local grasp resistance is the best one can hope for in grasping systems whose contacts react according to pre-specified mechanical contact laws. When such contact laws are implemented as fingertip feedback control laws, local grasp resistance is equivalent to the notion of *disturbance rejection* from control theory.

6.3 Duality of Immobilizing and Resistant Grasps under Frictionless Contact Conditions

As this part of the book will focus on secure grasps in the case of *frictionless* contacts, let us establish duality between wrench resistant and immobilizing grasps under frictionless contact conditions. The grasp resistance condition associated with frictionless contacts has a simple c-space interpretation. A frictionless finger force can be parametrized as $f_i = \lambda_i n_i$, where $\lambda_i \geq 0$ is the force magnitude and n_i is $\mathcal{B}$'s inward unit normal at x_i. The wrench induced by this force, $\boldsymbol{w}_i = \lambda_i(n_i, x_i \times n_i)$, is collinear with the finger c-obstacle outward normal at q_0, $\eta_i(q_0) = (n_i, x_i \times n_i)$. Substituting $\boldsymbol{w}_i = \lambda_i \eta_i(q_0)$ in (6.4) gives the grasp resistance condition:

$$\{\lambda_1 \eta_1(q_0) + \cdots + \lambda_k \eta_k(q_0) \,:\, \lambda_1, \ldots, \lambda_k \geq 0\} = T^*_{q_0}\mathcal{C} \quad m = 3 \text{ or } 6.$$

The grasp resistance condition is thus equivalent to the requirement that the finger c-obstacle normals positively span the object's wrench space.

Next consider the complementary approach, which seeks to prevent the object's free motions by proper placement of the finger bodies. The simplest type of immobility is based on preventing all possible instantaneous motions of the grasped object.

Definition 6 (**First-Order Immobilization**). *Let a rigid object $\mathcal{B}$ be held at an equilibrium grasp configuration q_0 by rigid and stationary fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The object is* **first-order immobilized** *when it has no free instantaneous motions at q_0,*

$$M_{1\ldots k} = \{\dot{q} \in T_{q_0}\mathbb{R}^m \,:\, \eta_i(q_0) \cdot \dot{q} \geq 0 \ \text{for } i = 1 \ldots k\} = \{0\},$$

where $T_{q_0}\mathcal{C}$ is $\mathcal{B}$'s tangent space at q_0.

The following theorem asserts that frictionless resistant grasps are equivalent to first-order immobilizing grasps (see appendix for a proof).

Theorem 2 (**Duality of first-order secure grasps**). *In the case of frictionless contacts, wrench resistant and first-order immobilizing grasps are* **dual** *notions—a frictionless equilibrium grasp is resistant iff it is first-order immobilizing.*

The theorem is based on the following notion of dual cones (see Figure 6.3). Let W be a wrench cone based at the origin of the object's wrench space, $T^*_{q_0}$. The *dual cone*, W^*, is the cone of tangent vectors given by

$$W^* = \{\dot{q} \in T_{q_0}\mathcal{C} : \boldsymbol{w} \cdot \dot{q} \leq 0 \ \text{ for all } \boldsymbol{w} \in W\}. \tag{6.6}$$

In our case W is the set of external wrenches that can be resisted by the finger wrenches, denoted $\mathcal{W}_{ext}$. The net wrench cone generated by the k fingers is given by $\mathcal{W} = \{\lambda_1 \eta_1(q_0) + \cdots + \lambda_k \eta_k(q_0) \,:\, \lambda_1, \ldots, \lambda_k \geq 0\}$, and $\mathcal{W}_{ext}$ is given by $\mathcal{W}_{ext} = \{-\boldsymbol{w} \,:\, \boldsymbol{w} \in \mathcal{W}\}$. It is shown in the appendix that the cone of free instantaneous motions, $M_{1\ldots k}$, is *dual* to $\mathcal{W}_{ext}$. This duality implies that $\mathcal{W}_{ext} = T^*_{q_0}\mathcal{C}$ iff $M_{1\ldots k} = \{0\}$, as illustrated in the following example.

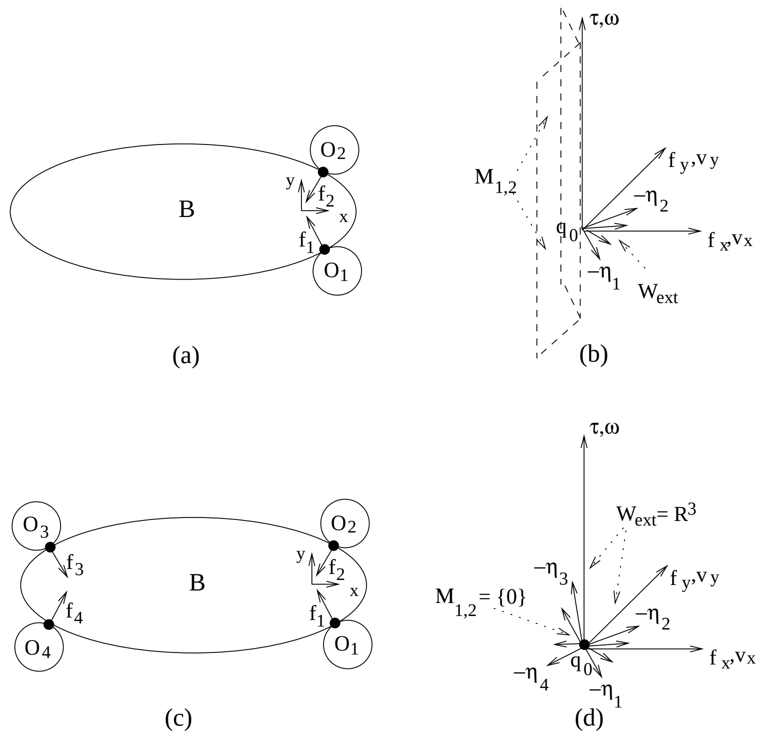

Figure 6.4: (a) A 2-finger grasp, and (b) the relation $M_{1,2} = (\mathcal{W}_{ext})^*$ for this grasp. (c) A 4-finger frictionless equilibrium grasp which is (d) resistant and first-order immobilizing.

Example—duality of $M_{1\ldots k}$ and $\mathcal{W}_{ext}$: Consider the two-finger grasp of an ellipse shown in Figure 6.3(a). The ellipse is located at a configuration q_0, and the finger forces f_1 and f_2 act along $\mathcal{B}$'s inward contact normals. When the frames F_W and F_B are selected at the contact normals intersection point, the finger wrenches are given by $\boldsymbol{w}_1 = (f_1, 0)$ and $\boldsymbol{w}_2 = (f_2, 0)$. These wrenches are collinear with the finger c-obstacle outward normals at q_0, $\eta_i(q_0) = \boldsymbol{w}_i$ for $i = 1, 2$. The wrench cone that can be resisted by the fingers, $\mathcal{W}_{ext} = \{\lambda_1(-\eta_1(q_0)) + \lambda_2(-\eta_2(q_0)) : \lambda_1, \lambda_2 \geq 0\}$, is the horizontal sector depicted in Figure 6.4(b). Its dual cone, $M_{1,2} = (\mathcal{W}_{ext})^*$, forms a three-dimensional wedge whose spine is aligned with the ω-axis in the object's tangent space, as depicted in Figure 6.4(b). Next consider the four-finger grasp of the ellipse shown in Figure 6.4(c). Now $\mathcal{W}_{ext}$ spans the entire wrench space as shown in Figure 6.4(d). Based on the duality property, $M_{1,2} = (\mathcal{W}_{ext})^* = \{0\}$, so this grasp is also a first-order immobilizing grasp.·

Based on Theorem 2, one can verify the safety of a candidate frictionless equilibrium grasp in two equivalent ways. Either one verifies the fingers ability to resists all external wrenches that might act on $\mathcal{B}$, or equivalently check that all the object's instantaneous motions are prevented by the fingers bodies.

6.4 Forward Look at Part II Chapters

This chapter provided an intuitive overview of the two fundamental approaches to grasp security. The remaining chapters of this part of the book will develop the analytical tools needed to analyze the mobility of an object held at a given grasp arrangement, then show how these tools can be used to address other important issues in grasp mechanics associated with frictionless contacts. Chapter 7 will develop the theory of *first-order* immobilizing grasps, which analyzes the first-order properties, or velocities, of the free motions of an object held by multiple fingers. Chapter 8 will extend the framework of Chapter 7 to consider the second-order properties of the free motions, and define *second-order* immobilizing grasps. Based on the methods introduced in Chapters 7 and 8, Chapter 9 will take up the following basic problem: *What is the minimum number of fingers required to immobilize arbitrary rigid objects*? A detailed analysis of this problem will be given for planar objects, where it will be seen that the minimal number of fingers depends on the finger geometry. A summary of the corresponding bounds for spatial objects will also be given. Chapter 10 will extend the immobilization analysis to include gravitational effects. In this case grasp safety amounts to ensuring that the fingers support the object at a *locally stable* equilibrium under the influence of gravity. The locally stable equilibria are precisely immobilizing grasps under the interpretation of gravity as a virtual "finger" acting at the object's center of mass. Therefore, the safety of a candidate grasp or posture under the influence of gravity can be judged with the immobilization tools developed in Chapters 7 and 8. The geometric techniques introduced in this part of the book will form the foundation for the subsequent analysis of frictional grasps in Part III of the book.

6.5 Bibliographical Notes

The notion of immobilizing grasps is traditionally called *form closure* in the grasping literature, though the traditional notion of form closure only covers what we term in this book as 1^{st}-order immobilization.. This notion dates to the work of Reuleaux in the 19^{th} century [3]. The notion of resistant grasps is traditionally called *force closure* in the grasping literature. While this notion is also deemed to originate with Reuleaux, the first formal discussion of force closure concepts can be found in the early 20^{th} century work of Somoff [5]. The study of form closure grasps in the modern robotics era starts with the work Lakshminarayana [2], while the modern analysis of robotic force closure grasps largely began with the work of Prof. Roth and the Ph.D. studies of Salisbury [4] and Kerr [1].

Appendix A: Proof Details

This appendix contains proofs of three key properties concerning immobilizing and resistant grasps. We begin with the property that every immobilizing grasp must form a frictionless equilibrium grasp.

Theorem 1. *Let a rigid object $\mathcal{B}$ be contacted by rigid and stationary fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. A necessary condition for $\mathcal{B}$ to be immobilized is that the fingers hold the object in a feasible* **frictionless equilibrium grasp.**

Proof: While the proposition holds true for general fingers, let us assume that the object is held by point fingers, $p_1 \ldots p_k$. Recall that the object's c-space is parametrized by

$q \in I\!R^{n+2}$, and that q_0 lies on the intersection of the c-obstacle boundaries associated with the k fingers. Now let $\eta_i \in I\!R^{n+2}$ be the unit outward normal to the i^{th} c-obstacle at q_0. The wrench induced by a normal force f_i at p_i is a positive multiple of η_i, $\boldsymbol{w}_i = \lambda_i \eta_i$ for $\lambda_i \geq 0$. At a k-finger equilibrium grasp the net wrench on the object must be zero:

$$\lambda_1 \eta_1 + \cdots + \lambda_k \eta_k = \vec{0} \qquad \lambda_1 \ldots \lambda_k \geq 0. \tag{6.7}$$

Let us now show that when the object is *not* held at an equilibrium grasp, it can simultaneously break contact with the k fingers and therefore is *not* immobilized. Let $\mathcal{W}$ be the collection of net wrenches affecting the object:

$$\mathcal{W} = \big\{ \boldsymbol{w} \in T^*_{q_0} I\!R^m \,:\, \boldsymbol{w} = \lambda_1 \eta_1 + \cdots + \lambda_k \eta_k \quad \text{for } \lambda_1 \ldots \lambda_k \geq 0 \big\}.$$

Since $\mathcal{W}$ is a positive linear combination of $\eta_1 \ldots \eta_k$, it forms a convex cone based at q_0. If the contact arrangement is not an equilibrium grasp, the cone $\mathcal{W}$ does not contain any full one-dimensional line passing through q_0 (such a line can only be generated by two opposing rays and is associated with an equilibrium grasp involving at least two fingers). A cone having this property is called *pointed* in convex analysis. The cone dual to $\mathcal{W}$ is given by $\mathcal{W}^* = \{h \in I\!R^m \,:\, h \cdot \boldsymbol{w} \leq 0 \text{ for all } \boldsymbol{w} \in \mathcal{W}\}$. The dual cone must have a non-empty interior, otherwise $(\mathcal{W}^*)^* = \mathcal{W}$ contains a full one-dimensional line. Every vector h from the interior of $\mathcal{W}^*$ satisfies $h \cdot \boldsymbol{w} < 0$ for all $\boldsymbol{w} \in \mathcal{W}$. Let $\dot{q} = -h$ be a tangent vector based at q_0, representing a particular instantaneous motion of the object which starts at q_0. Since $\eta_i \in \mathcal{W}$ for $i = 1 \ldots k$, $\dot{q}$ satisfies the inequality

$$\eta_i \cdot \dot{q} > 0 \quad \text{for } i = 1 \ldots k.$$

It follows that the c-space trajectory $q(t)$ such that $q(0) = q_0$ and $\dot{q}(0) = \dot{q}$ moves away from the k finger c-obstacles. Physically the object breaks contact with all k fingers along this motion, implying that the object is *not* immobilized by the k fingers. A frictionless equilibrium grasp is thus necessary for achieving form closure. □

The next proposition provides the basis for the property that every resistant grasp must form an equilibrium grasp (a frictionless equilibrium grasp in the case of frictionless contacts, a frictional equilibrium grasp in the case of frictional contacts).

Proposition 6.2.1. *Let a rigid object $\mathcal{B}$ be held by rigid fingers $\mathcal{O}_1, \ldots, \mathcal{O}_k$. The fingers form a feasible* **equilibrium grasp** *iff the net wrench cone, $\mathcal{W} = \mathcal{W}_1 + \cdots + \mathcal{W}_k$, contains a full subspace passing through the object's wrench space origin.*

Proof: The set product of the friction cones, $C_1 \times \cdots \times C_k$, forms a cone in the finger forces space $(f_1, \ldots, f_k)$. Since each friction cone C_i is a pointed cone,[3] $C_1 \times \cdots \times C_k$ is a pointed cone based at the origin of the space $(f_1, \ldots, f_k)$. Now pick a non-zero wrench in the subspace contained in $\mathcal{W}_1 + \cdots + \mathcal{W}_k$, $\boldsymbol{w}_0 \in T^*_{q_0} I\!R^m$. The grasp map, $G : C_1 \times \cdots \times C_k \to T^*_{q_0} I\!R^m$, maps the cone $C_1 \times \cdots \times C_k$ *onto* the net wrench cone. Hence there exists a non-zero vector of finger

[3] A cone C based at point x is a *pointed cone* when it does not contain any full line passing through its base point x.

forces, $\vec{f_0} \in C_1 \times \cdots \times C_k$, such that $G\vec{f_0} = \boldsymbol{w}_0$. Since vw_0 also lies in the subspace contained in $\mathcal{W}_1 + \cdots + \mathcal{W}_k$, there exists another non-zero finger force combination, $\vec{f}_0' \in C_1 \times \cdots \times C_k$, such that $G\vec{f}_0' = -\boldsymbol{w}_0$. Since G is a *linear map*, $G(\vec{f_0} + \vec{f}_0') = \boldsymbol{w}_0 - \boldsymbol{w}_0 = \vec{0}$. The finger forces $\vec{f_0} + \vec{f}_0'$ thus generate a zero net wrench on $\mathcal{B}$.

We now prove that $\vec{f_0} + \vec{f}_0'$ consists of non-zero forces which lie in the respective friction cones. Since $\vec{f_0}$ and $\vec{f}_0'$ lie in the cone $C_1 \times \cdots \times C_k$, their sum, $\vec{f_0} + \vec{f}_0'$, also lies in this cone. Therefore $\vec{f_0} + \vec{f}_0'$ consists of allowed finger forces. Since $C_1 \times \cdots \times C_k$ is a pointed cone, $\vec{f_0} + \vec{f}_0'$ must be a non-zero vector of finger forces; otherwise $C_1 \times \cdots \times C_k$ contains two opposing rays collinear with $\vec{f_0}$ and $-\vec{f_0}$, and therefore is not a pointed cone. It follows that $\vec{f_0} + \vec{f}_0'$ consists of permissible non-zero forces. $\square$

The last property is the duality between resistant and first-order immobilizing grasps.

Theorem 2. *In the case of frictionless contacts, resistant and first-order immobilizing grasps are* **dual** *notions—a frictionless equilibrium grasp is resistant iff it is first-order immobilizing.*

Proof: The net wrench cone generated by the k fingers is given by $\mathcal{W} = \{\lambda_1 \eta_1(q_0) + \cdots + \lambda_k \eta_k(q_0) \,:\, \lambda_1, \ldots, \lambda_k \geq 0\}$. The set of external wrenches that can be resisted by these fingers wrenches, $\mathcal{W}_{ext}$, is given by

$$\mathcal{W}_{ext} = \{-\boldsymbol{w} \,:\, \boldsymbol{w} \in \mathcal{W}\} = \{\lambda_1(-\eta_1(q_0)) + \cdots + \lambda_k(-\eta_k(q_0)) \,:\, \lambda_1, \ldots, \lambda_k \geq 0\}\,.$$

The set of $\mathcal{B}$'s instantaneous free motions at q_0 is obtained by intersecting the first-order free halfspaces (see Section 6.1),

$$M_{1\ldots k} = \cap_{i=1}^{k} M_i = \{\dot{q} \in T_{q_0}I\!R^m \,:\, \eta_i(q_0) \cdot \dot{q} \geq 0 \text{ for } i = 1 \ldots k\}.$$

Since $\boldsymbol{w}_{ext} = \lambda_1(-\eta_1(q_0)) + \cdots + \lambda_k(-\eta_k(q_0))$ for all $\boldsymbol{w}_{ext} \in \mathcal{W}_{ext}$, we obtain the following relation between $M_{1\ldots k}$ and $\mathcal{W}_{ext}$,

$$\forall \boldsymbol{w}_{ext} \in \mathcal{W}_{ext} \;\; \forall \dot{q} \in M_{1\ldots k} \qquad \boldsymbol{w}_{ext} \cdot \dot{q} = \sum_{i=1}^{k} \lambda_i(-\eta_i(q_0)) \cdot \dot{q} \leq 0.$$

Based on the definition of dual cone in eq. 6.6, $(\mathcal{W}_{ext})^* = \{\dot{q} : \boldsymbol{w} \cdot \dot{q} \leq 0 \text{ for all } \boldsymbol{w} \in \mathcal{W}_{ext}\} = M_{1\ldots k}$. The cone of free instantaneous motions, $M_{1\ldots k}$, is thus *dual* to the cone of resistible external wrenches, $\mathcal{W}_{ext}$. Therefore $\mathcal{W}_{ext} = T^*_{q_0}I\!R^m$ iff $M_{1\ldots k} = \{0\}$. $\square$

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