A Brief Introduction to Algebraic Systems

1 Groups

Definition 1.1: A *Group* is a non-empty set $\mathcal{G}$, along with a binary operation, $*$, such that for $a, b, c \in \mathcal{G}$,

G-1) $a * b \in \mathcal{G}$ (closure)

G-2) $(a * b) * c = a * (b * c)$ (associative)

G-3) There exists a unique element $e \in \mathcal{G}$ such that $a * e = e * a = a$, for every $a \in \mathcal{G}$. (identity)

G-4) For every $a \in \mathcal{G}$, there exists a unique element $a^{-1} \in \mathcal{G}$ such that $a * a^{-1} = a^{-1} * a = e$. (inverse)

Definition 1.2: A group $\mathcal{G}$ is said to be *abelian* or (*commutative*) if $a * b = b * a$, for all $a, b \in \mathcal{G}$.

Remark 1.1: A non-empty set $\mathcal{S}$ is called a semi-group if the binary operation "$*$" satisfies axioms G-1 and G-2 only.

2 Rings

Definition 2.1 A *Ring* is a non-empty set $\mathcal{R}$ with two binary operations: "+" and "$*$", such that

R-1) $\mathcal{R}$ forms an abelian group under the operation +, i.e., for all $a, b, c \in \mathcal{R}$,

- i) $a + b \in \mathcal{R}$
- ii) $a + b = b + a$
- iii) $(a + b) + c = a + (b + c)$
- iv) There is a unique identity element $0 \in \mathcal{R}$ such that $a + 0 = a$ for every $a \in \mathcal{R}$.
- v) There exists a unique $-a \in \mathcal{R}$ such that $a + (-a) = 0$ for every $a \in \mathcal{R}$.

R-2) $\mathcal{R}$ forms a semi-group under the operation $*$, i.e.,

- i) $a * b \in \mathcal{R}$
- ii) $(a * b) * c = a * (b * c)$

R-3) The operation $*$ is distributive with respect to $+$, i.e., for $a, b, c \in \mathcal{R}$,

$$\begin{aligned} a * (b + c) &= a * b + a * c \\ (b + c) * a &= b * a + c * a. \end{aligned}$$

Definition 2.2

- A *Commutative Ring* is a ring $\mathcal{R}$ that satisfies the commutative law with respect to the operation "$*$".
- A *Ring with Unity* is a ring $\mathcal{R}$ that has an identity element e with respect to the operation "$*$".
- A *Division Ring* or (*Skew Field*) is a ring with all its non-zero elements forming a group under "$*$" operation, i.e., there exists an inverse $a^{-1} \in \mathcal{R}$ for all $a \neq 0,\ a \in \mathcal{R}$.

3 Fields

Definition 3.1: A *Field* $\mathcal{K}$ is a commutative ring in which set of non-zero elements form a group under "$*$" operation.

In other words, a field $\mathcal{K}$ is an abelian group with 0 as its identity under "$+$" operation, and $\mathcal{K} - \{0\}$ forms an abelian group with e as its identity under the "$*$" operation satisfying the distributive law: $a * (b + c) = a * b + a * c$ and $(a + b) * c = a * c + b * c$ for all $a, b, c \in \mathcal{K}$.

4 Vector Spaces

Definition 4.1: A non-empty set $\mathcal{V}$ is said to be a *Vector Space* over a field $\mathcal{K}$ if it consists of a set of elements termed "vectors" and two binary operations: "$\oplus$", a vector addition, and "$\cdot$", a scalar multiplication, such that for $\vec{u}, \vec{v}, \vec{w} \in \mathcal{V}$ and $\alpha, \beta \in \mathcal{K}$,

V-1) $\vec{u} \oplus \vec{v} \in \mathcal{V}$ (closure)

V-2) $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ (commutative)

V-3) For all $\vec{u} \in \mathcal{V}$, there exists a unique $\vec{0} \in \mathcal{V}$ such that $\vec{u} \oplus \vec{0} = \vec{u}$.

V-4) $\vec{u} \oplus (-\vec{u}) = \vec{0}$ for every $\vec{u} \in \mathcal{V}$.

V-5) $(\vec{u} \oplus \vec{v}) \oplus \vec{w} = \vec{u} \oplus (\vec{v} \oplus \vec{w})$ (associative)

V-6) $\alpha \cdot \vec{u} \in \mathcal{V}$

V-7) $\alpha \cdot (\beta \cdot \vec{u}) = (\alpha \cdot \beta) \cdot \vec{u}$ ("$\cdot$" is associative).

V-8) $e \cdot \vec{u} = \vec{u}$ for all $\vec{u} \in \mathcal{V}$, where e is the identity element of $\mathcal{K}$ under "$*$".

V-9) $\alpha \cdot (\vec{u} \oplus \vec{v}) = \alpha \cdot \vec{u} + \alpha \cdot \vec{v}$

V-10) $(\alpha + \beta) \cdot \vec{u} = \alpha \cdot \vec{u} + \beta \cdot \vec{u}$

Remark 4.1: The vector space $\mathcal{V}$ forms an abelian group under the vector addition $\oplus$.

Remark 4.2 An n-dimensional vector spave $\mathcal{V}$ over a field $\mathcal{K}$ consists of n-tuples of elements in the field $\mathcal{K}$, which can be written as

$$\begin{aligned} \vec{u} &= (u_1, u_2, \ldots, u_n)^T \\ \vec{v} &= (v_1, v_2, \ldots, v_n)^T, \quad \forall\, \vec{u}, \vec{v} \in \mathcal{V} \end{aligned}$$

where $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathcal{K}$.

The vector addition, "$\oplus$", is defined by

$$\begin{aligned} \vec{u} \oplus \vec{v} &= (u_1, u_2, \ldots, u_n) \oplus (v_1, v_2, \ldots v_n) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) \end{aligned}$$

where the "+" operator is the "+" operator in $\mathcal{K}$.

And the identity for vector addition $\oplus$ is $\vec{0} = (0, \ldots, 0)^T$. The scalar multiplication, "$\cdot$", is defined by

$$\alpha \cdot \vec{u} = (\alpha * u_1, \alpha * u_2, \ldots, \alpha * u_n)^T, \qquad \text{for} \quad \alpha \in \mathcal{K}$$

where the "$*$" operator is the "$*$" operator in $\mathcal{K}$. One can verify that the vector addition and scalar mutiplication defined above satisfy axioms V-1) to V-10). We usually write $\mathcal{V} = \mathcal{K}^n$ for this case, where $\mathcal{K}$ is usually $\mathbf{R}$ or $\mathbf{C}$.

5 Algebras

Definition 5.1: An *Algebra* over a field $\mathcal{K}$, is a set $\mathcal{A}$, which is a vector space over $\mathcal{K}$ along with a vector multiplication, $\otimes$, such that for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{A}$ and $\lambda \in \mathcal{K}$,

A-1) $\mathbf{a} \otimes \mathbf{b} \in \mathcal{A}$

A-2) $\lambda \cdot (\mathbf{a} \otimes \mathbf{b}) = (\lambda \cdot \mathbf{a}) \otimes \mathbf{b} = \mathbf{a} \otimes (\lambda \cdot \mathbf{b})$

A-3) $\mathbf{a} \otimes (\mathbf{b} \oplus \mathbf{c}) = \mathbf{a} \otimes \mathbf{b} \oplus \mathbf{a} \otimes \mathbf{c}$ and $(\mathbf{a} \oplus \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes \mathbf{c} \oplus \mathbf{b} \otimes \mathbf{c}$

Remark 5.1: If $(\mathbf{a} \otimes \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes (\mathbf{b} \otimes \mathbf{c})$ holds for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}$, then $\mathcal{A}$ is called an associative algebra.

6 References:

1. Herstein, I.N., *Topics in Algebra*, 2ed, John Wiley and Sons, 1975.

2. Fraleigh, J.B., *A First Course in Abstract Algebra*, 3ed, Addison-Wesley, 1982.