1 The Differential Geometry of Curves

This section reviews some basic definitions and results concerning the differential geometry of curves. These results will be immediately applicable to the analysis of planar bodies, whose boundaries can be represented by curves. The next section will discuss the analogous notions for the surfaces that bound three-dimensional bodies.

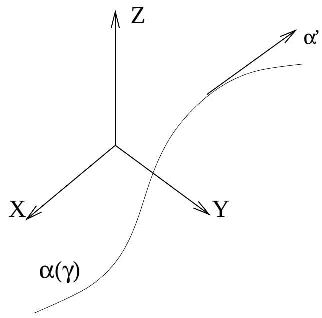

Figure 1: Schematic diagram of 3-dimensional curve

We wish to formally investigate the properties of curves, such as the one seen in Figure 1.

Definition 1 *Let $I = (a, b)$ be an open interval in $\mathbb{R}$. A* parametrized differential curve *is a differentiable map $\alpha\colon\ I \rightarrow \mathbb{R}^3$.*

That is, for each $\gamma \in I$, $\alpha(\gamma) = [x(\gamma)\ y(\gamma)\ z(\gamma)]^T$. The parameter γ is called the "curve parameter," and it provides a measure of distance along the curve α.

Definition 2 *The vector*

$$\frac{d\alpha(\gamma)}{d\gamma} = \alpha^{'}(\gamma) = [x^{'}(\gamma)\ y^{'}(\gamma)\ z^{'}(\gamma)]^T$$

(where a $'$ indicates differentiation) is the tangent *to the curve α at γ. A point γ^* such that $\alpha^{'}(\gamma^*) = 0$ is called a* singular point *of the curve. A parametrized differential curve $\alpha\colon\ I \rightarrow \mathbb{R}^3$ is said to be* regular *if $\alpha^{'}(\gamma) \neq 0$ for all $\gamma \in I$.*

Definition 3 *The* arc-length *of a regular curve $\alpha\colon\ I \rightarrow \mathbb{R}^3$ is defined as*

$$s(\gamma) = \int_a^{\gamma} |\alpha^{'}(\sigma)| d\sigma \tag{1}$$

If α is regular, then $\frac{ds}{d\gamma} = |\alpha^{'}(\sigma)|$

The tangent with respect to the arc-length parameter is:

$$\mathbf{u}(s) = \frac{d\alpha}{ds} = \frac{d\alpha}{d\gamma}\frac{d\gamma}{ds} = \frac{1}{|\alpha^{'}|}\alpha^{'}.$$

That is, the arc-length tangent vector always has unit length. Therefore, arc-length measures how far one progresses along the curve by moving at constant unit velocity.

Definition 4 *If $|\alpha^{'}| = 1$ for all $\gamma \in I$, then α is said to be* arc-length parametrized. *If α is regular, then it can always be reparametrized by arc-length (using Equation (1)).*

1.1 Curvature

Let $\alpha(s)$ be an arc-length parametrized curve. Let $\mathbf{v}$ be a unit length vector that depends upon a real parameter, t: $\mathbf{v}^T(t)\mathbf{v}(t) = 1$. Noting that

$$\frac{d}{dt}\mathbf{v}^T(t)\mathbf{v}(t) = 2\mathbf{v}^T(t)\mathbf{v}^{'}(t) = 0$$

it is easily seen that the derivative of a unit length vector, $\mathbf{v}$, is a vector, $\mathbf{v}^{'}$, that is orthogonal to $\mathbf{v}$. Thus

$$\frac{d\alpha^{'}(s)}{ds} = \alpha^{''}(s) \perp \alpha^{'}(s).$$

The norm of $\alpha^{''}(s)$ indicates how quickly the tangent vector $\mathbf{u}(s) = \alpha^{'}(s)$ is changing direction with respect to local variations in s.

Definition 5 *The quantity $|\alpha^{''}(s)|$ is called the* curvature *of $\alpha(s)$ at s, and is denoted by $\kappa(s)$. If $\kappa(s) \neq 0$, then*

$$\mathbf{n}(s) = \frac{1}{\kappa(s)}\alpha^{''}(s) \tag{2}$$

is the unit length normal vector *at s.*

The curvature function effectively measures how much the curve is bending at each point s. In order to physically interpret this bending, we introduce the following notions.

Definition 6 *The* osculating plane *(see Figure 2) at $\alpha(s)$ is the plane containing the point $\alpha(s)$ and which is spanned by the tangent and normal vectors. The* osculating circle *at $\alpha(s)$ is a circle with radius $R = 1/\kappa(s)$ whose center lies along a line collinear with the normal vector at $\alpha(s)$ a distance R from $\alpha(s)$.*

In other words, the curvature function $\kappa(s)$ measures the bend of the curve $\alpha(s)$ in the osculating plane. If we interpret the osculating circle as a curve, then the osculating circle and the curve $\alpha(s)$ have the same first and second derivatives at $\alpha(s)$.

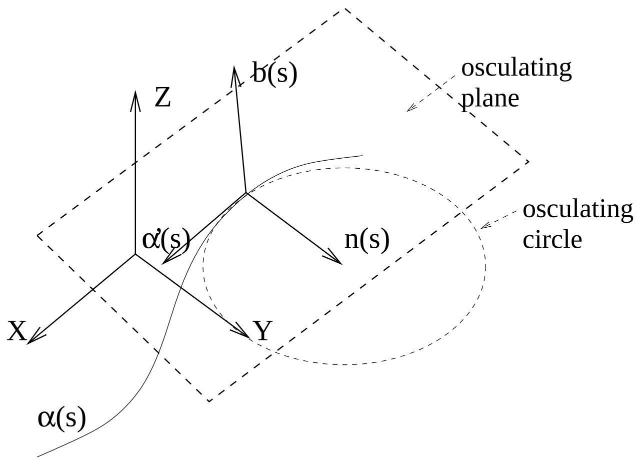

Figure 2: The osculating plane and circle of a 3-dimensional curve

Definition 7 *The unit vector* $\mathbf{b}(s) = \alpha'(s) \times \mathbf{n}(s)$ *is the curve's* binormal *vector at* $\alpha(s)$.

For a regular curve, at each s where $\kappa(s) \neq 0$, we have a unique set of orthonormal vectors: the tangent, normal, and binormal vectors. The *Frenet-Serret* frame is the right-handed coordinate frame whose origin is located at $\alpha(s)$ and whose basis vectors are $\alpha'(s)$, $\mathbf{n}(s)$, and $\mathbf{b}(s)$.

1.2 Torsion

The binormal vector is always orthogonal to the osculating plane. Therefore, the derivative of the binormal vector with respect to arc-length measures how fast the osculating plane changes as s varies.

$$\begin{aligned} \mathbf{b}'(s) &= \frac{d}{ds}(\alpha'(s) \times \mathbf{n}(s)) = \alpha''(s) \times \mathbf{n}(s) + \alpha'(s) \times \mathbf{n}'(s)) \\ &= \kappa(s)(\mathbf{n}(s) \times \mathbf{n}(s)) + \alpha'(s) \times \mathbf{n}'(s)) = \alpha'(s) \times \mathbf{n}'(s) \end{aligned}$$

From this equation we can see that $\mathbf{b}'(s)$ must be orthogonal to the tangent vector, $\alpha'(s)$. Since $\mathbf{b}$ is a unit vector, $\mathbf{b}'(s)$ must also be orthogonal to $\mathbf{b}$. Hence, $\mathbf{b}'(s)$ must be parallel (or anti-parallel) to the normal vector, $\mathbf{n}(s)$:

$$\mathbf{b}'(s) = \tau(s)\mathbf{n}(s)$$

where the proportionality constant, $\tau(s)$, is termed the *torsion* of $\alpha(s)$. Torsion measures the rate at which osculating plane twists about the normal vector with respect to variation in s. Note that if a curve lies strictly in a plane, then $\tau(s) = 0$ for all s.

An expression for $\tau(s)$ can be derived as follows.

$$\tau(s) = \tau(s)\mathbf{n}(s) \cdot \mathbf{n}(s) = \mathbf{n}(s) \cdot \mathbf{b}'(s)$$

$$
\begin{aligned}
&= \frac{1}{\kappa(s)}\alpha''(s)\cdot(\alpha'(s)\times\mathbf{n}'(s)) \\
&= \frac{1}{\kappa(s)}\alpha''(s)\cdot\left[\alpha'(s)\times\left(\frac{1}{\kappa(s)}\alpha''(s)\right)'\right] \\
&= \frac{1}{\kappa^3(s)}\alpha''(s)\cdot\left[\alpha'(s)\times\left(\kappa(s)\alpha'''(s)-\kappa'(s)\alpha''(s)\right)\right]
\end{aligned}
$$

However, since $\alpha''\cdot(\alpha'\times\alpha'')=0$, the above equation reduces to

$$
\tau(s)=\frac{1}{\kappa^2(s)}\alpha''(s)\cdot(\alpha'(s)\times\alpha'''(s))
$$

1.3 The Frenet Formulas

We have already derived formulas that describe the variations of the tangent vector and binormal vector with respect to variations in s. The variation in the normal vectors can be determined as follows:

$$
\begin{aligned}
\mathbf{n}'(s) &= \frac{d}{ds}(\mathbf{b}(s)\times\mathbf{u}(s))=\mathbf{b}'(s)\times\mathbf{u}(s)+\mathbf{b}(s)\times\mathbf{u}'(s) \\
&= \tau(s)(\mathbf{n}(s)\times\mathbf{u}(s))+\mathbf{b}(s)\times(\kappa(s)\mathbf{n}(s))=-(\kappa(s)\mathbf{u}(s)+\tau(s)\mathbf{b}(s))
\end{aligned}
$$

Consequently, the basis vectors of the Frenet-Serret frame vary as follows:

$$
\begin{aligned}
\mathbf{u}'(s) &= \kappa(s)\mathbf{n}(s) &(3)\\
\mathbf{n}'(s) &= \tau(s)\mathbf{b}(s)-\kappa(s)\mathbf{u}(s) &(4)\\
\mathbf{b}'(s) &= \tau(s)\mathbf{n}(s) &(5)
\end{aligned}
$$

The orientation of the Frenet-Serret Frame can be described by the rotation matrix whose columns are its basis vectors:

$$
R(s)=\begin{pmatrix}\mathbf{u}(s) & \mathbf{n}(s) & \mathbf{b}(s)\end{pmatrix}
$$

The Frenet formulas can then be expressed as:

$$
\frac{d}{ds}R(s)=R(s)\begin{pmatrix}0 & -\kappa(s) & 0\\ \kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}
$$

Based on these equations, standard existence and uniqueness results from the theory of differential equations can be used to prove the following theorem.

Theorem 1 The Fundamental Theorem of the Local Theory of Curves. *Given functions $\kappa(s) \neq 0$ and $\tau(s)$ for $s \in I = (a, b)$, there exists are regular parametrized curve α: $I \to \mathbb{R}^3$ such that s is the arc-length, $\kappa(s)$ is the curvature, and $\tau(s)$ is the torsion of this curve α. Moreover, any other curve, α_2, satisfying the same conditions differs from α by a rigid motion (i.e., a rigid translation and rotation of the curve).*

This theorem states that a regular curve can be uniquely (up to rigid body displacement) reconstructed from its curvature and torsion functions. Hence, the curvature and torsion functions are intrinsic properties of a curve. When the curve is restricted to a plane, this theorem means that a regular planar curve can be uniquely defined, up to a rigid body planar displacement, by its curvature function.