The Damped Pseudo Inverse

The "damped" pseudo-inverse addressess the problem of the possible discontinuity of the pseudo-inverse at a singular configuration. The key idea behind the damped pseudo-inverse is to find the joint velocities, $\dot{\vec{\theta}}$, that minimize the quantity:

$$||J_{ST}(\vec{\theta})\dot{\vec{\theta}} - V_{ST}||^2 + \rho^2||\dot{\vec{\theta}}||^2$$

where $\rho << 1$ is a "damping factor," J_{ST} is the manipulator Jacobian matrix, and V_{ST} is the tool frame velocity. Hereafter, J_{ST} will be abbreviated to J. The particular choice of hybrid, body, or spatial coordinates is arbitrary, though it is most logical to pose this problem in hybrid coordinates.

It can be shown that the solution to this problem is:

$$\dot{\vec{\theta}} = J_\rho^\dagger V_{ST}$$

where $J_\rho^\dagger$ is the "damped pseudo-inverse:"

$$J_\rho^\dagger = J^T(JJ^T + \rho^2 I)^{-1}.$$

Note that as $\rho \to 0$, the damped pseudo-inverse is identical to the Moore-Penrose Pseudo-Inverse.

We can analyze the effect of this damping term using the singular value decomposition. Let the Jacobian matrix have the decomposition:

$$J = U\Sigma V^T$$

Then the damped pseudo-inverse is:

$$\begin{aligned} J_\rho^\dagger &= J^T(JJ^T + \rho^2 I)^{-1} = V\Sigma^T U^T(U\Sigma V^T V\Sigma^T U^T + \rho^2 I)^{-1} \\ &= V\Sigma^T U^T[U(\Sigma\Sigma^T + \rho^2 I)U^T]^{-1} \\ &= V\Sigma^T U^T U(\Sigma\Sigma^T + \rho^2 I)^{-1}U^T \\ &= V\Sigma^T(\Sigma\Sigma^T + \rho^2 I)^{-1}U^T \\ &= V\Sigma_\rho U^T \end{aligned}$$

where:

$$\Sigma_\rho = \begin{bmatrix} \tilde{\sigma}_1 & 0 & \dots & 0 \\ 0 & \tilde{\sigma}_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \tilde{\sigma}_m \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad \text{where} \quad \tilde{\sigma}_j = \frac{\sigma_j}{\sigma_j^2 + \rho^2}$$

That is, $J_\rho^\dagger$ has singular values $\tilde{\sigma}_j = \frac{\sigma_j}{\sigma_j^2+\rho^2}$, where σ_j is the j^{th} singular value of the Jacobian matrix. Here again, $\tilde{\sigma}_j \to 1/\sigma_j$ as $\rho \to 0$.

What happens near a singularity? To analyze this behavior first note that in general a SVD can be expressed as:

$$J = U\Sigma V^T = \sum_{i=1}^{m} \sigma_i \vec{u}_i \vec{v}_i^T$$

where $\vec{u}_i$ and $\vec{v}_i$ are the i^{th} columns of the matrices U and V. To understand the behavior of the damped pseudo-inverse near a singularity, let's consider the maximum value of the ratio $||\dot{\vec{\theta}}||/||V_{ST}||$ for $||V_{ST}|| = 1$ when the damped pseudo-inverse is used to determine the manipulator's joint velocities.

$$\max_{||V_{ST}||=1} \frac{||\dot{\vec{\theta}}||}{||V_{ST}||} = \max_{||V_{ST}||=1} \frac{||J_\rho^\dagger V_{ST}||}{||V_{ST}||} = \max_{||V_{ST}||=1} ||\sum_{i=1}^{m} \left(\frac{\sigma_i}{\sigma_i^2+\rho^2}\right) \vec{v}_i \vec{u}_i^T V_{ST}|| \qquad (1)$$

Since the matrices U and V are orthogonal, they only rotate vectors, but do not elongate them. Hence, the maximum value of Equation 1 will occur when V_{ST} aligns with the direction of the maximum singular value of $J_\rho^\dagger$. Hence, the maximum joint velocity that the manipulator mechanism undergoes will be a function of the maximum singular value of $J_\rho^\dagger$ as it nears a singularity.

How can we determine the relationship between the choice of the damping factor ρ and the maximum joint velocity? Let $\sigma_i = \gamma\rho$. I.e, gamma is a convenient scaling factor. Thus,

$$\tilde{\sigma}_i = \frac{\sigma_i}{\sigma_i^2+\rho^2} = \frac{1}{\rho}\left(\frac{\gamma}{1+\gamma^2}\right)$$

Consider the following limiting cases:

- If $\gamma << 1$ (i.e. the singular values of $J_\rho^\dagger$ are small), then

$$\tilde{\sigma}_i \simeq \frac{\gamma}{\rho} << \frac{1}{\rho}$$

- If $\gamma \simeq 1$, then

$$\tilde{\sigma}_i \simeq \frac{1}{2}\left(\frac{1}{\rho}\right)$$

- If $\gamma >> 1$, then

$$\tilde{\sigma}_i \simeq \frac{1}{\gamma}\left(\frac{1}{\rho}\right) << \frac{1}{\rho}$$

A more rigorous analysis shows that $||\dot{\vec{\theta}}||/||V_{ST}||$ is bounded by $(1/2)\rho^{-1}$. For large σ_i (i.e., J is well condition since it is far from a singularity), then $\tilde{\sigma}_i \simeq \sigma_i^{-1}$. Hence, away from singularities, the damping term has little effect on the pseudo-inverse. However, near a singularity, the singular value of the damped pseudo inverse does not go toward infinity, but towards the limiting value of $(1/2)\rho^{-1}$. However, the price to pay is deviation of the tool frame from the desired trajectory.