ME 115(a): Solutions to the Final Exam

Problem #1 (30 points)

Part (a): (5 points) Assume that the base frame is chosen to be coincident with the link #1 reference frame when $\theta_1 = 0$. Further assume that the tool frame is the same as link frame #3. Then, the Denavit-Hartenberg parameters are:

$$\begin{array}{llll} a_0 = 0; & \alpha_0 = 0; & d_1 = 0; & \theta_1 = \text{ variable} \\ a_1 \neq 0; & \alpha_1 = 0; & d_2 = 0; & \theta_2 = \text{ variable} \\ a_2 \neq 0; & \alpha_2 = 0; & d_3 = \text{ variable}; & \theta_3 = 0 \end{array}$$

Part (b): (10 points) The forward kinematic equations are: $g_{S,T} = g_{S,1} g_{1,2} g_{2,3}$, where:

$$g_{S,1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{1,2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{2,3} = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{S,3} = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & d_3 \end{bmatrix}$$

where $c_1 = \cos\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$, etc.

Part (c): (15 Points)

We immediate see from the forward kinematics that:

$$z_T = d_3.$$

The (x, y) location of the tool frame origin is completely specified by the geometry of the first two links, which look just like the 2R planar manipulator problem that was discussed in class. You could immediately use that solution, or devise a variety of solutions. Here is one approach. Squaring and summing the x and y-coordinates of the tool frame origin yields:

$$x_T^2 + y_T^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\theta_2)$$

which can be solved for θ_2 (two solutions):

$$\theta_2 = -\theta_2' = \cos^{-1}\left[\frac{x_T^2 + y_T^2 - a_1^2 - a_2^2}{2a_1 a_2}\right].$$

There are a variety of ways to determine θ_1. Note that from the expressions for the (x, y) tool frame origin we get:

$$x_T - a_1 c_1 = a_2 c_{12} \qquad y_T - a_1 s_1 = a_2 s_{12}$$

Squaring and summing both equations:

$$x_T^2 + y_T^2 + a_1^2 - a_2^2 - 2a_1(x_T c_1 + y_T s_1) = 0 \tag{1}$$

Using the identities:

$$\cos\theta_1 = \frac{1-q^2}{1+q^2} \qquad \sin\theta_1 = \frac{2q}{1+q^2}$$

where $q = \tan(\theta_1/2)$, one can rewrite Eq. (1) as a quadratic in q.

Problem #2: (15 points)

Part (a): (10 points)

1. $\tilde{g}$ will be orthogonal if $\tilde{g}^T\tilde{g} = I$.

$$\begin{aligned}\tilde{g}^T\tilde{g} &= [R+\epsilon(\hat{p}R)]^T[R+\epsilon(\hat{p}R)] == [R^T - \epsilon R^T\hat{p}][R+\epsilon\hat{p}R] \\ &= R^TR + \epsilon R^T\hat{p}R - \epsilon R^T\hat{p}R - \epsilon^2 R^T\hat{p}^2R = I\end{aligned}$$

2. Let $\tilde{g}_1 = [R_1 + \epsilon\hat{p}_1R_1]$ and $\tilde{g}_2 = [R_2 + \epsilon\hat{p}_2R_2]$. Then:

$$\begin{aligned}\tilde{g}_3 = \tilde{g}_1\tilde{g}_2 &= [R_1+\epsilon\hat{p}_1R_1][R_2+\epsilon\hat{p}_2R_2] \\ &= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2) + \epsilon^2(\hat{p}_1R_1\hat{p}_2R_2) \\ &= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2)\end{aligned}$$

   Note that $g_3 = g_1g_2$ is given by:

$$g_3 = \begin{bmatrix} R_1R_2 & R_1\overline{p}_2 - \overline{p}_1 \\ \overline{0}^T & 1 \end{bmatrix}$$

   Hence, $\tilde{g}_3 = R_1R_2 + \epsilon((\widehat{R_1\overline{p}_2} - \overline{p}_1)R_1R_2)$. Note that:

$$\begin{aligned}((\widehat{R_1\overline{p}_2} - \overline{p}_1)R_1R_2) &= (R_1\hat{p}_2R_1^T - \hat{p}_1)R_1R_2 \\ &= R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2\end{aligned}$$

   Hence, the two are equivalent.

3. Let $\tilde{\xi}_1 = \overline{\omega}_1 + \epsilon\overline{V}_1$ and $\tilde{\xi}_2 = \overline{\omega}_2 + \epsilon\overline{V}_2$. Then:

$$\begin{aligned}\tilde{\xi}_1\tilde{\xi}_2 &= (\overline{\omega}_1 + \epsilon\overline{V}_1)\cdot(\overline{\omega}_2 + \epsilon\overline{V}_2) \\ &= \overline{\omega}_1\cdot\overline{\omega}_2 + \epsilon(\overline{V}_1\cdot\overline{\omega_2} + \overline{\omega}_1\cdot\overline{V}_2) + \epsilon^2(\overline{V}_1\cdot\overline{V}_2) \\ &= \overline{\omega}_1\cdot\overline{\omega}_2 + \epsilon(\overline{V}_1\cdot\overline{\omega_2} + \overline{\omega}_1\cdot\overline{V}_2)\end{aligned}$$

   The dual part is the reciprocal product.

Part(b): (5 points)

1. First note that if transformation g consists of rotation R and displacement $\overline{p}$, then:

$$Ad_g\xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \overline{V} \\ \overline{\omega} \end{bmatrix} = \begin{bmatrix} R\overline{V} + \hat{p}R\overline{\omega} \\ R\overline{\omega} \end{bmatrix}$$

The "dual" version of this vector is $R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega})$. But, $\tilde{g}\tilde{\xi} = R + \epsilon(\hat{p}R)$ and $\tilde{\xi} = \overline{\omega} + \epsilon\overline{V}$. Hence:

$$\begin{aligned} \tilde{g}\tilde{\xi} &= (R + \epsilon(\hat{p}R))(\overline{\omega} + \epsilon\overline{V}) && (2) \\ &= R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega}) + \epsilon^2(\hat{p}R\overline{V}) && (3) \\ &= R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega}) && (4) \end{aligned}$$

Thus, the two are equivalent.

2. Let $\xi_1 = [\overline{V}_1^T \ \overline{\omega}_1^T]$ and $\xi_2 = [\overline{V}_2^T \ \overline{\omega}_2^T]$. Then:

$$\begin{aligned} \tilde{\xi}_1 \cdot \tilde{\xi}_2 &= (\overline{\omega}_1 + \epsilon\overline{V}_1) \cdot (\overline{\omega}_2 + \epsilon\overline{V}_2) = \overline{\omega}_1 \cdot \overline{\omega}_2 + \epsilon(\overline{V}_1 \cdot \overline{\omega}_2 + \overline{\omega}_1 \cdot \overline{V}_2) + \epsilon^2(\overline{V}_1 \cdot \overline{V}_2) \\ &= \overline{\omega}_1 \cdot \overline{\omega}_2 + \epsilon(\overline{V}_1 \cdot \overline{\omega}_2 + \overline{\omega}_1 \cdot \overline{V}_2) \end{aligned}$$

The dual part of this, $\overline{V}_1 \cdot \overline{\omega}_2 + \overline{\omega}_1 \cdot \overline{V}_2$, is the reciprocal product of ξ_1 and ξ_2.

Problem #3: (20 points)

Each finger applies a "wrench" to the disk object due to its contact with the disk. Since we are assuming a frictionless contact, the finger can only apply forces to the disk that are normal to the disk's boundary. Hence, each finger applies a pure force in the direction of the boundary normal vector, which corresponds to a zero pitch screw.

Define a coordinate system whose origin lies at the common intersection of all of the finger forces at the center of the disk. Choose the z-axis of this system to be normal to the plane of the disk. Let the x-axis coincide with one of the finger contact normals. Thus, the screw coordinates for the three wrenches are:

$$\xi_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(120^o) \\ -\sin(120^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} -\cos(240^o) \\ -\sin(240^o) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If the disk is not immobilized, there there must exist a twist (i.e., an instantaneous motion of the disk) that is **reciprocal** to the finger wrenches. The $\xi_R = [0\ 0\ 1\ 0\ 0\ 0]^T$ denote the zero pitch twist that corresponds to rotation of the disk about a vertical axis passing through the origin of the reference frame (i.e., the concurrency point of the three contact normals). This twist is reciprocal to each of the finger wrenches, and therefore the fingers **can not** stop any rotational motions of the disk. Hence, the object is not immobilized.

Problem #4: (20 points)

Assign a coordinate system whose origin is located at a point C, where the screw axis of S_1 intersects the horizontal plane, P. Let the z-axis be collinear with the positive S_1 direction, and let the x-axis be collinear with the mutually perpendicular line between S_1 and S_2. In this coordinate system, the screw coordinates of S_1 and S_2 are (recalling that $h_1 = h_2 = 0$ and that the screw axes are separated by a distance a):

$$\xi_1 = \begin{bmatrix} h_1\overline{\omega}_1 + \overline{\rho}_1 \times \overline{\omega}_1 \\ \overline{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} h_2\overline{\omega}_2 + \overline{\rho}_2 \times \overline{\omega}_2 \\ \overline{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -a \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We require that any screw, S_R, which is reciprocal to both S_1 and S_2 also lie in the plane P. Let's divide the analysis into those screws that have finite pitch and those that potentially have infinite pitch.

We can parametrize all finite pitch screws that lie in P as follows:

$$\xi_R = \begin{bmatrix} h_R\overline{\omega}_R + \overline{\rho_R} \times \overline{\omega}_R \\ \overline{\omega}_R \end{bmatrix}$$

where h_R is the pitch of the reciprocal screw while $\overline{\omega}_R$ is a unit length vector collinear with the screw axis of the reciprocal screw, $\overline{\rho}_R$ is a vector from the origin of the reference frame described above to a point on the reciprocal screw axis. By assumption, both $\overline{\omega}_R$ and $\overline{\rho}_R$ must also lie in P. We can describe any screw that lies in the plane by two scalars: d (the distance along the mutually perpendicular line between S_1 and S_R) and θ, the angle between the mutually perpendicular line and the x-axis of the reference coordinate system that was defined above. In terms of these scalars:

$$\overline{\omega}_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \overline{\rho}_R = d \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and hence:

$$\xi_R = \begin{bmatrix} -h_R\sin\theta \\ h_R\cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

If S_R is reciprocal to S_1, then the reciprocal product between these two screws must be zero. Letting $\circ$ denote the reciprocal product,

$$\xi_1 \circ \xi_R = d = 0$$

This implies that the screw axis of S_R must intersect the axis of S_1. If S_R is reciprocal to S_2, then:

$$-a\cos\theta = 0.$$

Since $a \neq 0$ by assumption, it must be true that $\cos\theta = 0$, or $\theta = \pm\pi/2$. This implies that the axis of the reciprocal screw is parallel to x-axis of the coordinate system. Hence, all finite pitch reciprocal screws must simultaneously intersect the axes of S_1 and S_2. They can have any pitch value.

Note that if the pitch of the reciprocal screw is infinite, then it is automatically reciprocal to both screws.

Problem #5: (15 points)

Part (a): The homogeneous displacement is most easily computed via a similarity transform. That is, let g_R denote the displacement which is pure rotation about the pole:

$$g_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and g_D denote the displacment from the reference frame to a frame whose origin is at the pole, and with basis vectors parallel to the reference frame basis vectors:

$$g_D = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the displacement of the rigid body is:

$$g = g_D g_R g_D^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & p_x(1-\cos\theta) + p_y\sin\theta \\ \sin\theta & \cos\theta & -p_x\sin\theta + p_y(1-\cos\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

In other words, the displacement vector is $\overline{d} = (\mathbf{I} - \mathbf{A})\overline{p}$.

Part (b): Noting that:

$$\begin{aligned} \cos\theta &= \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = Z_4^2 - Z_3^2 \\ \sin\theta &= 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = 2Z_3Z_4 \end{aligned}$$

then:

$$\begin{aligned} d_x &= p_x(1-\cos\theta) + p_y\sin\theta \\ &= 2[p_x\sin^2\tfrac{\theta}{2} + p_y\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}] \\ &= 2[-Z_2Z_3 + Z_1Z_4] \\ d_y &= -p_x\sin\theta + p_y(1-\cos\theta) \\ &= 2[-p_x\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} + p_y\sin^2\tfrac{\theta}{2}] \\ &= 2[Z_2Z_4 + Z_1Z_3] \\ \theta &= 2Atan2[Z_3, Z_4] \end{aligned}$$