ME 115(a): Final Exam Solution
2000/2001

Problem 1: (35 Points)

The "trick" to this problem is how to orient the manipulator in its "home" position in order to make the analysis straightforward. See Fig. 1 for one appropriate way to do this. In this case, it was necessary to redefine the positive direction of the third joint axis.

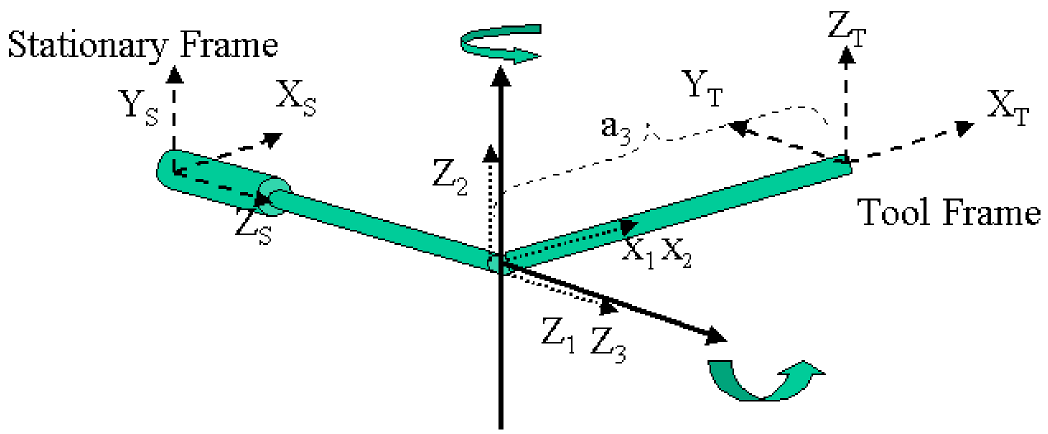

Figure 1: Schematic of PRR Manipulator in "home" position

Part (a) (Denavit-Hartenberg parameters): (3 points) Assuming the definitions shown in Figure 1, the parameters are:

$$\begin{array}{cccc} a_0 = 0 & \alpha_0 = 0 & d_1 = \text{variable} & \theta_1 = 0 \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = 0 & \theta_3 = \text{variable} \\ a_3 = L & \alpha_3 = -\frac{\pi}{2} & d_4 = 0 & \theta_4 = 0 \end{array} \tag{1}$$

Part (b) (forward kinematics): (7 points) Using the Denavit-hartenberg approach:

$$g_{S1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_1 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$g_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_1 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{3T} = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The total forward kinematics is:

$$g_{ST} = g_{S1}\ g_{12}\ g_{23}\ g_{3T} \ = = \begin{bmatrix} c_2c_3 & -s_2 & c_2s_3 & Lc_2c_3 \\ s_3 & 0 & c_3 & Ls_3 \\ -s_2c_3 & -c_2 & s_2s_3 & d_1 - Ls_2c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

where $c_j = \cos\theta_j$, and $s_j = \theta_j$.

Part (c) (inverse kinematics): (15 points) Let (x_D, y_D, z_D) denote the desired location of the origin of the tool frame, with respect to th origin of the stationary frame. From part (b), we know that:

$$x_D = Lc_2c_3 \tag{5}$$
$$y_D = Ls_3 \tag{6}$$
$$z_D = d_1 - Ls_2c_3. \tag{7}$$

From Eq. (6) we see that $\sin\theta_3 = \frac{y_D}{L}$. Assuming that $|y_D/L| \leq 1$, we have that

$$\theta_3 = \sin^{-1}\left[\frac{y_D}{L}\right]. \tag{8}$$

Two solutions for θ_3 can be obtained from Eq. (8). The second solution, dubbed θ_3', is $\theta_3' = \pi - \theta_3$. From Eq. (5) we have two θ_2 solutions for each θ_3 solution:

$$\theta_2 - \cos^{-1}\left[\frac{x_D}{L\cos\theta_3}\right]. \tag{9}$$

The second solution is $\theta_2' = -\theta_2$. Finally, from Eq. (7), we can solve for d_1 for each given (θ_2, θ_3) pair:

$$d_1 = z_D + L\sin\theta_2\cos\theta_3.$$

Part (d) (Jacobian matrix): (10 points)

The spatial Jacobian has the form:

$$J = \begin{bmatrix} \overline{\xi}_1 & \overline{\xi}_2^{'} & \overline{\xi}_3^{'} \end{bmatrix} \tag{10}$$

where

$$\begin{aligned} \overline{\xi}_2^{'} &= Ad_{e^{d_1 \hat{\xi}_1}} \overline{\xi}_2 & (11) \\ \overline{\xi}_3^{'} &= Ad_{e^{d_1 \hat{\xi}_1}} Ad_{e^{\theta_2 \hat{\xi}_2}} \overline{\xi}_3 & (12) \end{aligned}$$

Recalling the form of a twist for a prismatic joint, and noting that $d_1 = 0$ in the home position, simple observation of Fig. 1, leads to:

$$\overline{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \overline{\xi}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \overline{\xi}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{13}$$

Note that:

$$Ad_{e^{d_1 \hat{\xi}_1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -d_1 & 0 \\ 0 & 1 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad Ad_{e^{\theta_2 \hat{\xi}_2}} = \begin{bmatrix} c_2 & 0 & s_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -s_2 & 1 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -s_2 & 0 & c_2 \end{bmatrix} \tag{14}$$

Substitution of Eq.s (14) and (13) into Eq.s (11), (12), and (10) yields:

$$J_{ST}^{s} = \begin{bmatrix} 0 & -d_1 & 0 \\ 0 & 0 & d_1 \sin\theta_2 \\ 1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta_2 \end{bmatrix} \tag{15}$$

Problem 2: (15 points)

Part (a): (10 points)

1. $\tilde{g}$ will be orthogonal if $\tilde{g}^T \tilde{g} = I$.

$$\begin{aligned} \tilde{g}^T \tilde{g} &= [R + \epsilon(\hat{p}R)]^T [R + \epsilon(\hat{p}R)] = [R^T - \epsilon R^T \hat{p}][R + \epsilon \hat{p} R] \\ &= R^T R + \epsilon R^T \hat{p} R - \epsilon R^T \hat{p} R - \epsilon^2 R^T \hat{p}^2 R = I \end{aligned}$$

2. Let $\tilde{g}_1 = [R_1 + \epsilon \hat{p}_1 R_1]$ and $\tilde{g}_2 = [R_2 + \epsilon \hat{p}_2 R_2]$. Then:

$$\begin{aligned}\tilde{g}_3 = \tilde{g}_1 \tilde{g}_2 &= [R_1 + \epsilon \hat{p}_1 R_1][R_2 + \epsilon \hat{p}_2 R_2] \\ &= R_1 R_2 + \epsilon(R_1 \hat{p}_2 R_2 + \hat{p}_1 R_1 R_2) + \epsilon^2(\hat{p}_1 R_1 \hat{p}_2 R_2) \\ &= R_1 R_2 + \epsilon(R_1 \hat{p}_2 R_2 + \hat{p}_1 R_1 R_2)\end{aligned}$$

Note that $g_3 = g_1 g_2$ is given by:

$$g_3 = \begin{bmatrix} R_1 R_2 & R_1 \overline{p}_2 - \overline{p}_1 \\ \overline{0}^T & 1 \end{bmatrix}$$

Hence, $\tilde{g}_3 = R_1 R_2 + \epsilon((\widehat{R_1 \overline{p}_2 - \overline{p}_1}) R_1 R_2)$. Note that:

$$\begin{aligned}((\widehat{R_1 \overline{p}_2 - \overline{p}_1}) R_1 R_2) &= (R_1 \hat{p}_2 R_1^T - \hat{p}_1) R_1 R_2 \\ &= R_1 \hat{p}_2 R_2 + \hat{p}_1 R_1 R_2\end{aligned}$$

Hence, the two are equivalent.

3. Let $\tilde{\xi}_1 = \overline{\omega}_1 + \epsilon \overline{V}_1$ and $\tilde{\xi}_2 = \overline{\omega}_2 + \epsilon \overline{V}_2$. Then:

$$\begin{aligned}\tilde{\xi}_1 \tilde{\xi}_2 &= (\overline{\omega}_1 + \epsilon \overline{V}_1) \cdot (\overline{\omega}_2 + \epsilon \overline{V}_2) \\ &= \overline{\omega}_1 \cdot \overline{\omega}_2 + \epsilon(\overline{V}_1 \cdot \overline{\omega_2} + \overline{\omega}_1 \cdot \overline{V}_2) + \epsilon^2(\overline{V}_1 \cdot \overline{V}_2) \\ &= \overline{\omega}_1 \cdot \overline{\omega}_2 + \epsilon(\overline{V}_1 \cdot \overline{\omega_2} + \overline{\omega}_1 \cdot \overline{V}_2)\end{aligned}$$

The dual part is the reciprocal product.

Part(b): (5 points)

1. First note that if transformation g consists of rotation R and displacement $\overline{p}$, then:

$$Ad_g \xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \overline{V} \\ \overline{\omega} \end{bmatrix} = \begin{bmatrix} R\overline{V} + \hat{p}R\overline{\omega} \\ R\overline{\omega} \end{bmatrix}$$

The "dual" version of this vector is $R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega})$. But, $\tilde{g}\tilde{\xi} = R + \epsilon(\hat{p}R)$ and $\tilde{\xi} = \overline{\omega} + \epsilon \overline{V}$. Hence:

$$\begin{aligned}\tilde{g}\tilde{\xi} &= (R + \epsilon(\hat{p}R))(\overline{\omega} + \epsilon \overline{V}) && (16) \\ &= R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega}) + \epsilon^2(\hat{p}R\overline{V}) && (17) \\ &= R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega}) && (18)\end{aligned}$$

Thus, the two are equivalent.

2. Let $\xi_1 = [\overline{V}_1^T \ \overline{\omega}_1^T]$ and $\xi_2 = [\overline{V}_2^T \ \overline{\omega}_2^T]$. Then:

$$\begin{aligned}\tilde{\xi}_1 \cdot \tilde{\xi}_2 &= (\overline{\omega}_1 + \epsilon \overline{V}_1) \cdot (\overline{\omega}_2 + \epsilon \overline{V}_2) = \overline{\omega}_1 \cdot \overline{\omega}_2 + \epsilon(\overline{V}_1 \cdot \overline{\omega}_2 + \overline{\omega}_1 \cdot \overline{V}_2) + \epsilon^2(\overline{V}_1 \cdot \overline{V}_2) \\ &= \overline{\omega}_1 \cdot \overline{\omega}_2 + \epsilon(\overline{V}_1 \cdot \overline{\omega}_2 + \overline{\omega}_1 \cdot \overline{V}_2)\end{aligned}$$

The dual part of this, $\overline{V}_1 \cdot \overline{\omega}_2 + \overline{\omega}_1 \cdot \overline{V}_2$, is the reciprocal product of ξ_1 and ξ_2.

Problem 3: (10 points) Problem 8 of chapter 3

Let g_ST denote the forward kinematics map for a given tool frame T. Suppose there is a different tool frame, T', which is rigidly displaced from T by $g_{TT'}$. I.e., $g_{TT'}$ is a constant matrix. The spatial velocity is

$$\hat{V}^s_{ST} = \dot{g}_{ST} g^{-1}_{ST}. \tag{19}$$

The spatial velocity associated to the alternative tool frame location is

$$\hat{V}^s_{ST'} = \dot{g}_{ST'} g^{-1}_{ST'} = \frac{d}{dt}(g_{ST} g_{TT'})(g_{ST} g_{TT'})^{-1} = (\dot{g}_{ST} g_{TT'})(g^{-1}_{TT'} g^{-1}_{ST}) = \dot{g}_{ST} g^{-1}_{ST} = \hat{V}_{ST} \tag{20}$$

Problem 4: (20 points) Problem 18(a,b) of Chapter 2

Part (a): By definition, $v^b = R^T \dot{p}$, $\omega^b = (R^T \dot{R})^\vee$, and $\omega^s = (\dot{R} R^T)^\vee$. Consequently,

$$\begin{aligned} \overline{V}^h &= \begin{bmatrix} \dot{p} \\ \omega^s \end{bmatrix} = \begin{bmatrix} R R^T \dot{p} \\ (R R^T \dot{R} R^T)^\vee \end{bmatrix} \\ &= \begin{bmatrix} R(R^T \dot{p}) \\ ((R(R^T \dot{R})^\vee)\hat{)}^\vee \end{bmatrix} = \begin{bmatrix} R v^b \\ R\omega^b \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} v^b \\ \omega^b \end{bmatrix} \end{aligned}$$

Part (b): There are many ways to solve this problem. For example, you could either start with Proposition 2.14 or Proposition 2.15 on page 59 of MLS which relate the velocities of three frames, A, B, and C. Let's choose Prop. 2.15:

$$V^b_{ac} = Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc} \tag{21}$$

Using the fact that

$$V^h_{ac} = \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{ac}$$

Eq. (21) can be written as:

$$\begin{aligned} V^h_{ac} &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} (Ad_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}) \\ &= \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} \begin{bmatrix} R^T_{bc} & -R^T_{bc}\hat{p}_{bc} \\ 0 & R^T_{bc} \end{bmatrix} V^b_{ab} + \begin{bmatrix} R_{ac} & 0 \\ 0 & R_{ac} \end{bmatrix} V^b_{bc} \\ &= \begin{bmatrix} R_{ab} & -R_{ab}\hat{p}_{bc} \\ 0 & R_{ab} \end{bmatrix} V^b_{ab} + \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{bc} & 0 \\ 0 & R_{bc} \end{bmatrix} V^b_{bc} \\ &= \begin{bmatrix} I & -(\widehat{R_{ab} p_{bc}}) \\ 0 & R_{ab} \end{bmatrix} \begin{bmatrix} R_{ab} & 0 \\ 0 & R_{ab} \end{bmatrix} V^b_{ab} + Ad_{R_{ab}} V^h_{bc} \\ &= Ad_{-R_{ab} p_{bc}} V^h_{ab} + Ad_{R_{ab}} V^h_{bc} \end{aligned} \tag{22}$$

Problem 5: (20 Points) Problem 11(d,e) in Chapter 2 of the MLS text.

Part (d): Let

$$g = \begin{bmatrix} A & \overline{p} \\ \overline{0}^T & 1 \end{bmatrix}$$

where $A \in SO(2)$ and $\overline{p} \in \mathbb{R}^2$. Then direct calculation shows that $\dot{g}g^{-1}$ and $g^{-1}\dot{g}$ are twists. The spatial and body velocities have definitions analogous to those for 3-dimensional rigid bodies

Part (e): Let $\hat{V}^b$ denote the planar body velocity:

$$\hat{V}^b = \begin{bmatrix} \hat{\omega}^b & \overline{v}^b \\ \overline{0}^T & 0 \end{bmatrix}$$

where $\hat{\omega}^b \in so(2)$, $\overline{v}^b \in \mathbb{R}^2$. Then the planar spatial velocity is:

$$\begin{aligned} \hat{V}^s &= Ad_g \hat{V}^b = g\hat{V}^b g^{-1} \\ &= \begin{bmatrix} R & \overline{p} \\ \overline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega}^b & \overline{v}^b \\ \overline{0}^T & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\overline{p} \\ \overline{0}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} R\hat{\omega}^b R^T & -R\hat{\omega}^b R^T \overline{p} + R\overline{v}^b \\ \overline{0}^T & 0 \end{bmatrix} \end{aligned}$$

Therefore:

$$\hat{\omega}^s = R\hat{\omega}^b R^T \qquad \overline{v}^s = R\overline{v}^b - R\hat{\omega}^b R^T \overline{p} = R\overline{v}^b - \hat{\omega}^s \overline{p}$$

The spatial angular velocity can be simplified as follows:

$$\hat{\omega}^s = R\hat{\omega}^b R^T = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} = \omega \begin{bmatrix} 0 & -\det(R) \\ \det(R) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega}^b$$

Using this result:

$$\overline{v}^s = R\overline{v}^b - \hat{\omega}^s \overline{p} = R\overline{v}^b + \omega^b \begin{bmatrix} p_y \\ -p_x \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \end{bmatrix} \begin{bmatrix} \overline{v}^b \\ \omega^b \end{bmatrix}$$

Therefore:

$$V^s = \begin{bmatrix} \overline{v}^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ \overline{0}^T & 1 \end{bmatrix} V^b$$