ME 115(a): Final Exam Solution
2001/2002

Problem 1: (35 Points)

Part (a): (3 points). Assume that the base frame is chosen to be coincident with the link #1 reference frame when $\theta_1 = 0$. Further assume that the tool frame is the same as link frame #3. Then, the Denavit-Hartenberg parameters are:

$$\begin{array}{llll} a_0 = 0; & \alpha_0 = 0; & d_1 = 0; & \theta_1 = \text{ variable} \\ a_1 \neq 0; & \alpha_1 = 0; & d_2 = 0; & \theta_2 = \text{ variable} \\ a_2 \neq 0; & \alpha_2 = 0; & d_3 = \text{ variable}; & \theta_3 = 0 \end{array}$$

Part (b): (7 points) The forward kinematic equations are: $g_{S,T} = g_{S,1}g_{1,2}g_{2,3}$, where:

$$g_{S,1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad g_{1,2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$g_{2,3} = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{S,3} = \begin{bmatrix} c_{12} & -s_{12} & 0 & (a_1c_1 + a_2c_{12}) \\ s_{12} & c_{12} & 0 & (a_1s_1 + a_2s_{12}) \\ 0 & 0 & 1 & d_3 \end{bmatrix}$$

where $c_1 = \cos\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$, etc.

Part (c): (15 Points)

We immediate see from the forward kinematics that:

$$z_T = d_3.$$

The (x, y) location of the tool frame origin is completely specified by the geometry of the first two links, which look just like the 2R planar manipulator problem that was discussed in class. You could immediately use that solution, or devise a variety of solutions. Here is one approach. Squaring and summing the x and y-coordinates of the tool frame origin yields:

$$x_T^2 + y_T^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\theta_2)$$

which can be solved for θ_2 (two solutions):

$$\theta_2 = -\theta_2^{'} = \cos^{-1}\left[\frac{x_T^2 + y_T^2 - a_1^2 - a_2^2}{2a_1a_2}\right].$$

There are a variety of ways to determine θ_1. Note that from the expressions for the (x, y) tool frame origin we get:

$$x_T - a_1 c_1 = a_2 c_{12} \qquad y_T - a_1 s_1 = a_2 s_{12}$$

Squaring and summing both equations:

$$x_T^2 + y_T^2 + a_1^2 - a_2^2 - 2a_1(x_T c_1 + y_T s_1) = 0 \tag{1}$$

Using the identities:

$$\cos\theta_1 = \frac{1-q^2}{1+q^2} \qquad \sin\theta_1 = \frac{2q}{1+q^2}$$

where $q = \tan(\theta_1/2)$, one can rewrite Eq. (1) as a quadratic in q.

Part (d): (10 points). The spatial Jacobian has the form:

$$J^s_{S,3} = \begin{bmatrix} \overline{\xi}_1 & \overline{\xi}'_2 & \overline{\xi}'_3 \end{bmatrix}$$

where the $\overline{\xi}_i$ are the twist coordinates of the joint axes (as described in the manipulator's stationary frame) in the zero configuration, and $\overline{\xi}'_2 = Ad_{e^{\theta_1\hat{\xi}_1}}\overline{\xi}_2$, $\overline{\xi}'_3 = Ad_{e^{\theta_1\hat{\xi}_1}}Ad_{e^{\theta_2\hat{\xi}_2}}\overline{\xi}_3$. If we choose the zero configuration with the first two links stretched out, then for the revolute joints (which correspond to zero pitch screws), the twist coordinates are:

$$\overline{\xi}_i = \begin{bmatrix} v_i \\ \overline{\omega}_i \end{bmatrix} = \begin{bmatrix} h_i\overline{\omega}_i + \overline{\rho}_i \times \overline{\omega}_i \\ \overline{\omega}_i \end{bmatrix} = \begin{bmatrix} \overline{\rho}_i \times \overline{\omega}_i \\ \overline{\omega}_i \end{bmatrix}$$

where $h_i = 0$ for revolute joints. In the stationary frame,

$$\overline{\omega}_i = \overline{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \overline{z}_S, \qquad \overline{\rho}_1 = \overline{0}, \qquad \overline{\rho}_2 = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} = a_1\overline{y}_S \tag{2}$$

where $\overline{y}_S$ and $\overline{z}_S$ are unit vectors along the y and z axes of the stationary frame. The prismatic joint corresponds to an infinite pitch screw (i.e., a pure translation) whose axis is vertical. Consequently,

$$\overline{\xi}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \overline{0} \\ \overline{z}_S \end{bmatrix}, \qquad \overline{\xi}_2 = \begin{bmatrix} 0 \\ -a_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a_1\overline{y}_S \\ \overline{z}_S \end{bmatrix}, \qquad \overline{\xi}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{z}_S \\ \overline{0} \end{bmatrix}.$$

Using Equation 2, we get

$$e^{\theta_1\hat{\xi}_1} = \begin{bmatrix} R_1 & \overline{0} \\ \overline{0}^T & 1 \end{bmatrix} \quad \text{where} \quad R_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consequently,

$$\overline{\xi}_2' = Ad_{e^{\theta_1 \hat{\xi}_1}} \overline{\xi}_2 = \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} -a_1 \overline{y}_S \\ \overline{z}_S \end{bmatrix} = \begin{bmatrix} a_1 \sin\theta_1 \\ -a_1 \cos\theta_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly,

$$e^{\theta_2 \hat{\xi}_2} = \begin{bmatrix} R_2 & \overline{p}_2 \\ \overline{0}^T & 1 \end{bmatrix} \quad \text{where} \quad R_1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\overline{p}_2$ turns out to be irrelevant. Consequently,

$$\overline{\xi}_3' = Ad_{e^{\theta_1 \hat{\xi}_1}} Ad_{e^{\theta_2 \hat{\xi}_2}} \overline{\xi}_3 = \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} R_2 & \hat{p}_2 R_2 \\ 0 & R_2 \end{bmatrix} \begin{bmatrix} \overline{z}_S \\ \overline{0} \end{bmatrix} = \begin{bmatrix} R_1 R_2 \overline{z}_S \\ \overline{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence:

$$J^s_{S,3} = \begin{bmatrix} 0 & a_1 \sin\theta_1 & 0 \\ 0 & -a_1 \cos\theta_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Problem 2: (15 points). The goal of this problem is two understand the possible motions of the object shown in Figure 1. Define a fixed reference frame whose origin lies at the midpoint between the two contact points, and whose y-axis aligns with the common line underlying the two contact normals. Intuitively, the two frictionless fingers can only apply forces along the y-axis of the reference frame. Hence, the object should be able to instantaneously slide along the x-axis, as well as rotate about the axis normal to the plane. We can show this formally as follows.

Since the grasped object is restricted to move in the plane, it has 3 degrees of freedom (DOF). All of its instantaneous motions can be expressed as a linear combination of three independent 1-DOF motions. There is not a unique choice for the basis vectors of this three dimensional set. Let us choose the basis vectors to be

$$\$_x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \$_y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \$_\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the twist vector of any velocity can be expressed as

$$V = c_x \$_x + c_y \$_y + c_\theta \$_\theta = \begin{bmatrix} c_x \\ c_y \\ 0 \\ 0 \\ 0 \\ c_\theta \end{bmatrix}$$

The forces that can be applied to the object by the frictionless finger contacts can be expressed as the wrench:

$$W = f_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + f_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $f_i > 0$ is the magnitude of the force applied by the i^{th} finger. The applied forces can not stop instantaneous motions which are reciprocal to the applied wrenches. The reciprocal product between the screw describing the wrench W and the velocity V is c_y. Hence, the fingers can only force c_y to be zero (i.e., stop motions along the y-axis). However, the can not stop translations along the x-axis and rotations.

Problem 3: (17 points). This problem extracted information about the rotation represented by this matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0.833333 & -0.186887 & 0.52022 \\ 0.52022 & 0.583333 & -0.623773 \\ -0.186887 & 0.79044 & 0.583333 \end{bmatrix}.$$

Part (a): (8 points). Compute the axis of rotation and the angle of rotation. The angle of rotation can be computed from the formula:

$$\cos\phi = \frac{r_{11} + r_{22} + r_{33} - 1}{2} = \frac{0.833333 + 0.583333 + 0.583333 - 1}{2} = 0.5$$

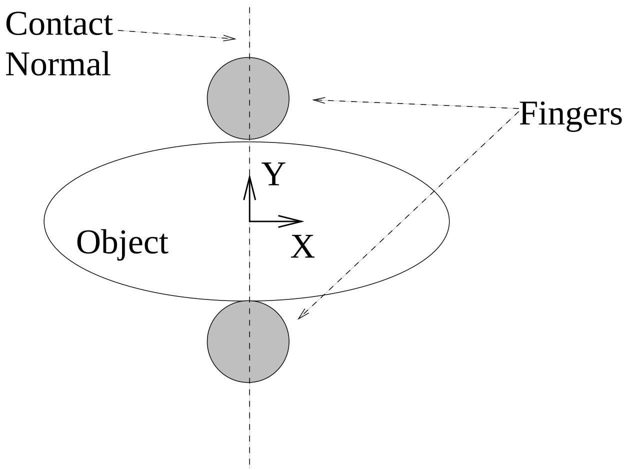

Figure 1: Schematic of two-fingered frictionless grasp

which yields solutions $\phi_1 = \cos^{-1}(0.5) = 60^o$ and $\phi_2 = -\phi_1 = -60^o$. The x, y, and z components of the unit vector axis of rotation can be found as:

$$\omega_x = \frac{r_{32} - r_{23}}{2\sin(60^o)} = \frac{0.79044 + 0.623773}{2 * 0.866025} = 0.816496 \tag{3}$$

$$\omega_y = \frac{r_{13} - r_{31}}{2\sin(60^o)} = \frac{0.52022 + 0.0.186887}{2 * 0.866025} = 0.408248 \tag{4}$$

$$\omega_z = \frac{r_{21} - r_{12}}{2\sin(60^o)} = \frac{0.52022 + 0.186887}{2 * 0.866025} = 0.408248 \tag{5}$$

Part (b): (4 points). The unit quarternion equivalent to this rotation is given by:

$$\begin{aligned} q &= (a, b, c, d) = (\cos(\frac{\phi}{2}), \omega_x \sin(\frac{\phi}{2}), \omega_y \sin(\frac{\phi}{2}), \omega_z \sin(\frac{\phi}{2})) \\ &= (0.866025, 0.408248, 0.204124, 0.204124) \end{aligned} \tag{6}$$

Part (c): (5 points). The z-y-z Euler were calculated in the class notes. If the successive angles are denoted ψ, ϕ, and γ, then:

$$\cos\phi = r_{33} \Rightarrow \phi = \cos^{-1}(0.58333) = 54.3147^o \tag{7}$$

$$\gamma = Atan2[\frac{r_{32}}{\sin\phi}, \frac{-r_{31}}{\sin\phi}] = Atan2[0.973169, 0.23009] = 76.6967^o \tag{8}$$

$$\psi = Atan2[\frac{r_{23}}{\sin\phi}, \frac{r_{13}}{\sin\phi}] = Atan2[-0.767973, 0.640481] = -50.1723^o \tag{9}$$

Problem 4: (20 points). (Problem 17(a,b) in Chapter 2 of MLS)

Part(a): (10 points). Let ξ_1 and ξ_2 denote the coordinates of two screws, as described in Frame A. Let frame B be displaced relative to frame A by g. Let η_1 and η_2 be the representation of these screws in Frame B. Hence:

$$\xi_1 = Ad_g\eta_1 \qquad \xi_2 = Ad_g\eta_2$$

Letting $\circ$ denote the reciprocal product:

$$\begin{aligned}
\eta_1 \circ \eta_2 \quad &= \eta_1^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \eta_2 \\
&= (Ad_{g^{-1}}\xi_1)^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} (Ad_{g^{-1}}\xi_2) \\
&= \xi_1^T \left(Ad_{g^{-1}}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Ad_{g^{-1}} \right) \xi_2
\end{aligned}$$

A direct calculation shows that

$$Ad_{g^{-1}}^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Ad_{g^{-1}} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

Hence, $\eta_1 \circ \eta_2 = \xi_1 \circ \xi_2$.

Part(b): (10 points). Using the same notation as in part (a):

$$\begin{aligned}
\eta_1 \cdot \eta_2 \quad &= \eta_1^T \eta_2 \\
&= (Ad_{g^{-1}}\xi_1)^T (Ad_{g^{-1}}\xi_2) \\
&= \xi_1^T Ad_{g^{-1}}^T Ad_{g^{-1}} \xi_2 \\
&= \xi_1^T \begin{bmatrix} I & -\hat{p} \\ \hat{p} & I - \hat{p}^2 \end{bmatrix} \xi_2 \neq \xi_1 \cdot \xi_2
\end{aligned}$$

Problem 5: (13 Points).

The Euler-Angle representation of the moving body's orientation is

$$R = R_\psi R_\phi R_\gamma$$

where R_ψ represents rotation by angle ψ about the body fixed z-axis, R_ϕ represents rotation by angle ϕ about the body fixed y-axis, and R_γ represents rotation by angle γ about the body fixed z-axis. The spatial angular velocity of the rigid body is:

$$\begin{aligned}
\hat{\omega}^s \quad &= \quad \dot{R}R^{-1} = (\dot{R}_\psi R_\phi R_\gamma + R_\psi \dot{R}_\phi R_\gamma + R_\psi R_\phi \dot{R}_\gamma) R_\gamma^{-1} R_\phi^{-1} R_\psi^{-1} \\
&= \quad \dot{R}_\psi R_\psi^{-1} + R_\psi (\dot{R}_\phi R_\phi^{-1}) R_\psi^{-1} + R_\psi R_\phi (\dot{R}_\gamma R_\gamma^{-1}) R_\phi^{-1} R_\psi^{-1} \\
&= \quad \hat{\omega}_\psi^s + R_\psi \hat{\omega}_\phi^s R_\psi^{-1} + R_\psi R_\phi \hat{\omega}_\gamma^s R_\phi^{-1} R_\psi^{-1} \qquad (10)
\end{aligned}$$

where $\omega_\psi^s = \dot{R}_\psi R_\psi^{-1}$, $\omega_\phi^s = \dot{R}_\phi R_\phi^{-1}$, $\omega_\gamma^s = \dot{R}_\gamma R_\gamma^{-1}$. Converting to vector form:

$$\overline{\omega}^s = (\hat{\omega}^s)^\vee = \overline{\omega}_\psi^s + R_\psi\, \overline{\omega}_\phi^s + R_\psi R_\phi\, \overline{\omega}_\gamma^s \qquad (11)$$

The angular velocities are simply:

$$\overline{\omega}^s_\psi = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \qquad \overline{\omega}^s_\phi = \begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix} \qquad \overline{\omega}^s_\gamma = \begin{bmatrix} 0 \\ 0 \\ \dot{\gamma} \end{bmatrix}$$

while the orientations of the frames are:

$$R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\phi = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_\gamma = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting into Equation (11) results in:

$$\overline{\omega}^s = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} -\dot{\phi}\sin\psi \\ \dot{\phi}\cos\psi \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\gamma}\cos\psi\sin\phi \\ \dot{\gamma}\sin\psi\sin\phi \\ \dot{\gamma}\cos\phi \end{bmatrix} = \begin{bmatrix} -\dot{\phi}\sin\psi + \dot{\gamma}\cos\psi\sin\phi \\ \dot{\phi}\cos\gamma + \dot{\gamma}\sin\psi\sin\phi \\ \dot{\psi} + \dot{\gamma}\cos\phi \end{bmatrix} \tag{12}$$