ME 115(a): Final Exam Solutions
(Winter Quarter 2005/2006)

Problem 1: (15 points)

The goal of this problem is two understand the possible motions of the object shown in Figure 1. Define a fixed reference frame whose origin lies at the midpoint between the two contact points, and whose y-axis aligns with the common line underlying the two contact normals (see Figure 1. Intuitively, the two frictionless fingers can only apply forces along the y-axis of the reference frame. Hence, the object should be able to instantaneously slide along the x-axis, as well as rotate about the axis normal to the plane. We can show this formally as follows.

Since the grasped object is restricted to move in the plane, it has 3 degrees of freedom (DOF). All of its instantaneous motions can be expressed as a linear combination of three independent 1-DOF motions. There is not a unique choice for the basis vectors of this three dimensional set. Let us choose the basis vectors to be

$$\$_x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \$_y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \$_\theta = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\$_x$ represents a unit velocity along the x-axis, $\$_y$ represents a unit velocity along the y-axis and $\$_\theta$ represents a unit angular velocity about the axis orthogonal to the plane. Thus, any velocity can be expressed as

$$V = c_x\$_x + c_y\$_y + c_\theta\$_\theta = \begin{bmatrix} c_x \\ c_y \\ 0 \\ 0 \\ 0 \\ c_\theta \end{bmatrix}$$

The forces that can be applied to the object by the frictionless finger contacts can be expressed as the wrench:

$$W = f_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + f_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ f_1 - f_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where $f_i > 0$ is the magnitude of the force applied by the i^{th} finger. The applied forces can not stop instantaneous motions which are reciprocal to the applied wrenches. Note that $\$_x$ and $\$_\theta$ are both reciprocal to the wrench of the finger forces. Hence, the fingers can not stop translations along the x-axis and rotations about the axis perpendicular to the plane.

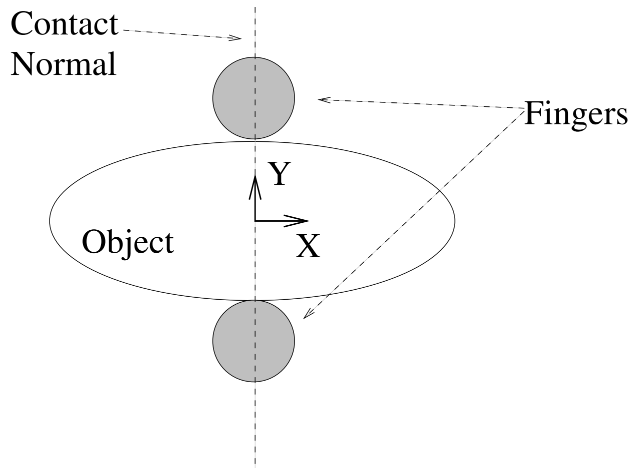

Figure 1: Schematic of two-fingered frictionless grasp

Problem 2: (15 points). The goal of this problem is to extract information about the physical rotation that is represented by this matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0.833333 & -0.186887 & 0.52022 \\ 0.52022 & 0.583333 & -0.623773 \\ -0.186887 & 0.79044 & 0.583333 \end{bmatrix}.$$

Part (a): (7 points). Compute the axis of rotation and the angle of rotation. The angle of rotation can be computed from the formula:

$$\cos\phi = \frac{r_{11} + r_{22} + r_{33} - 1}{2} = \frac{0.833333 + 0.583333 + 0.583333 - 1}{2} = 0.5$$

which yields solutions $\phi_1 = \cos^{-1}(0.5) = 60^o$ and $\phi_2 = -\phi_1 = -60^o$. The x, y, and z components of the unit vector axis of rotation can be found as:

$$\omega_x = \frac{r_{32} - r_{23}}{2\sin(60^o)} = \frac{0.79044 + 0.623773}{2 * 0.866025} = 0.816496 \tag{1}$$

$$\omega_y = \frac{r_{13} - r_{31}}{2\sin(60^o)} = \frac{0.52022 + 0.0.186887}{2 * 0.866025} = 0.408248 \tag{2}$$

$$\omega_z = \frac{r_{21} - r_{12}}{2\sin(60^o)} = \frac{0.52022 + 0.186887}{2 * 0.866025} = 0.408248 \tag{3}$$

Part (b): (4 points). The unit quarternion equivalent to this rotation is given by:

$$\begin{aligned} q &= (a, b, c, d) = (\cos(\frac{\phi}{2}), \omega_x \sin(\frac{\phi}{2}), \omega_y \sin(\frac{\phi}{2}), \omega_z \sin(\frac{\phi}{2})) \\ &= (0.866025, 0.408248, 0.204124, 0.204124) \end{aligned} \tag{4}$$

Part (c): (4 points). The z-y-z Euler were calculated in the class notes. If the successive angles are denoted ψ, ϕ, and γ, then:

$$\cos\phi = r_{33} \;\Rightarrow\; \phi = \cos^{-1}(0.58333) = 54.3147^o \tag{5}$$

$$\gamma = Atan2[\frac{r_{32}}{\sin\phi}, \frac{-r_{31}}{\sin\phi}] = Atan2[0.973169, 0.23009] = 76.6967^o \tag{6}$$

$$\psi = Atan2[\frac{r_{23}}{\sin\phi}, \frac{r_{13}}{\sin\phi}] = Atan2[-0.767973, 0.640481] = -50.1723^o \tag{7}$$

Problem 3: (20 Points)

This problem asked you to look at the represention and manipulation of spatial displacements using the concept of "dual numbers." A dual number, $\tilde{a}$, takes the form:

$$\tilde{a} = a_r + \epsilon\, a_d$$

where a_r is the "real" part of the dual number and a_d is the "dual" or "pure" part of the dual number. The bases for the dual numbers are 1 and ϵ, and they obey the rules:

$$\begin{aligned} 1\cdot 1 &= 1 \\ 1\cdot\epsilon &= \epsilon\cdot 1 = \epsilon \\ \epsilon^2 &= 0 \end{aligned}$$

Part (a): (10 points)

1. $\tilde{g}$ will be orthogonal if $\tilde{g}^T\tilde{g} = I$.

$$\begin{aligned} \tilde{g}^T\tilde{g} &= [R + \epsilon(\hat{p}R)]^T[R + \epsilon(\hat{p}R)] = [R^T - \epsilon R^T\hat{p}][R + \epsilon\hat{p}R] \\ &= R^TR + \epsilon R^T\hat{p}R - \epsilon R^T\hat{p}R - \epsilon^2 R^T\hat{p}^2R \\ &= I + \epsilon R^T(\hat{p} - \hat{p})R = I \end{aligned}$$

2. Let $\tilde{g}_1 = [R_1 + \epsilon\hat{p}_1R_1]$ and $\tilde{g}_2 = [R_2 + \epsilon\hat{p}_2R_2]$. Then:

$$\begin{aligned} \tilde{g}_3 = \tilde{g}_1\tilde{g}_2 &= [R_1 + \epsilon\hat{p}_1R_1][R_2 + \epsilon\hat{p}_2R_2] \\ &= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2) + \epsilon^2(\hat{p}_1R_1\hat{p}_2R_2) \\ &= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2) \end{aligned}$$

Note that $g_3 = g_1g_2$ is given by:

$$g_3 = \begin{bmatrix} R_1R_2 & R_1\overline{p}_2 - \overline{p}_1 \\ \overline{0}^T & 1 \end{bmatrix}$$

Hence, $\tilde{g}_3 = R_1R_2 + \epsilon((\widehat{R_1\overline{p}_2 - \overline{p}_1})R_1R_2)$. Note that:

$$\begin{aligned} ((\widehat{R_1\overline{p}_2 - \overline{p}_1})R_1R_2) &= (R_1\hat{p}_2R_1^T - \hat{p}_1)R_1R_2 \\ &= R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2 \end{aligned}$$

Hence, the two are equivalent.

3. Let $\tilde{\xi}_1 = \overline{\omega}_1 + \epsilon\overline{V}_1$ and $\tilde{\xi}_2 = \overline{\omega}_2 + \epsilon\overline{V}_2$. Then:

$$\begin{aligned}\tilde{\xi}_1\tilde{\xi}_2 &= (\overline{\omega}_1 + \epsilon\overline{V}_1)\cdot(\overline{\omega}_2 + \epsilon\overline{V}_2) \\ &= \overline{\omega}_1\cdot\overline{\omega}_2 + \epsilon(\overline{V}_1\cdot\overline{\omega_2} + \overline{\omega}_1\cdot\overline{V}_2) + \epsilon^2(\overline{V}_1\cdot\overline{V}_2) \\ &= \overline{\omega}_1\cdot\overline{\omega}_2 + \epsilon(\overline{V}_1\cdot\overline{\omega_2} + \overline{\omega}_1\cdot\overline{V}_2)\end{aligned}$$

The dual part is the reciprocal product.

Part(b): (5 points)

1. First note that if transformation g consists of rotation R and displacement $\overline{p}$, then:

$$Ad_g\xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R\end{bmatrix}\begin{bmatrix}\overline{V} \\ \overline{\omega}\end{bmatrix} = \begin{bmatrix} R\overline{V} + \hat{p}R\overline{\omega} \\ R\overline{\omega}\end{bmatrix}$$

The "dual" version of this vector is $R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega})$. But, $\tilde{g}\tilde{\xi} = R + \epsilon(\hat{p}R)$ and $\tilde{\xi} = \overline{\omega} + \epsilon\overline{V}$. Hence:

$$\begin{aligned}\tilde{g}\tilde{\xi} &= (R + \epsilon(\hat{p}R))(\overline{\omega} + \epsilon\overline{V}) && (8)\\ &= R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega}) + \epsilon^2(\hat{p}R\overline{V}) && (9)\\ &= R\overline{\omega} + \epsilon(R\overline{V} + \hat{p}R\overline{\omega}) && (10)\end{aligned}$$

Thus, the two are equivalent.

2. Let $\xi_1 = [\overline{V}_1^T\ \overline{\omega}_1^T]$ and $\xi_2 = [\overline{V}_2^T\ \overline{\omega}_2^T]$. Then:

$$\begin{aligned}\tilde{\xi}_1\cdot\tilde{\xi}_2 &= (\overline{\omega}_1 + \epsilon\overline{V}_1)\cdot(\overline{\omega}_2 + \epsilon\overline{V}_2) = \overline{\omega}_1\cdot\overline{\omega}_2 + \epsilon(\overline{V}_1\cdot\overline{\omega}_2 + \overline{\omega}_1\cdot\overline{V}_2) + \epsilon^2(\overline{V}_1\cdot\overline{V}_2) \\ &= \overline{\omega}_1\cdot\overline{\omega}_2 + \epsilon(\overline{V}_1\cdot\overline{\omega}_2 + \overline{\omega}_1\cdot\overline{V}_2)\end{aligned}$$

The dual part of this, $\overline{V}_1\cdot\overline{\omega}_2 + \overline{\omega}_1\cdot\overline{V}_2$, is the reciprocal product of ξ_1 and ξ_2.

Problem 4: (15 Points)

The geometry of this situation is recalled in Figure 2. S_1 is perpendicular to the plane, P, and has zero pitch: $h_1 = 0$. The screw axis of S_2 lies in P, and S_2 some non-zero pitch, h_2. The distance between S_1 and S_2, as measured along a mutually perpendicular line, is denoted a. The goal of this problem is to describe the set of all screws whose axes lie in P and that are reciprocal to both S_1 and S_2.

Assign a coordinate system whose origin is located at a point C, where the screw axis of S_1 intersects the horizontal plane, P, containing S_2. Let the z-axis be collinear with the positive S_1 direction, and let the x-axis be collinear with the mutually perpendicular line

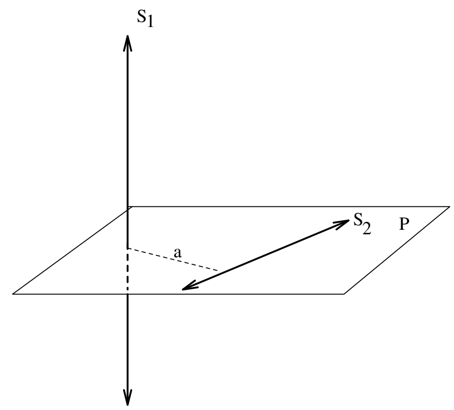

Figure 2: Two Screws

between S_1 and S_2. In this coordinate system, the screw coordinates of S_1 and S_2 are:

$$\xi_1 = \begin{bmatrix} h_1\overline{\omega}_1 + \overline{\rho}_1 \times \overline{\omega}_1 \\ \overline{\omega}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \xi_2 = \begin{bmatrix} h_2\overline{\omega}_2 + \overline{\rho}_2 \times \overline{\omega}_2 \\ \overline{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ h_2 \\ a \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

We require that any screw, S_R, which is reciprocal to both S_1 and S_2 also lie in the plane P. We can parametrize all screws that lie in P as follows:

$$\xi_R = \begin{bmatrix} h_R\overline{\omega}_R + \overline{\rho_R} \times \overline{\omega}_R \end{bmatrix}$$

where h_R is the pitch of the reciprocal screw while $\overline{\omega}_R$ is a unit length vector collinear with the screw axis of the reciprocal screw, $\overline{\rho}_R$ is a vector from the origin of the reference frame described above to a point on the reciprocal screw axis. By assumption, both $\overline{\omega}_R$ and $\overline{\rho}_R$ must also lie in P. We can describe any screw that lies in the plane by two scalars: d (the distance along the mutually perpendicular line between S_1 and S_R) and θ, the angle between the mutuallyperpendicular line and the x-axis of the reference coordinate system, which lies in P. In terms of these scalars:

$$\overline{\omega}_R = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \qquad \overline{\rho}_R = d \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

and hence:

$$\xi_R = \begin{bmatrix} -h_R\sin\theta \\ h_R\cos\theta \\ d \\ -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

If S_R is reciprocal to S_1, then the reciprocal product between these two screws must be zero. Letting $\circ$ denote the reciprocal product,

$$\xi_1 \circ \xi_R = d = 0$$

This implies that the screw axis of S_R must intersect the axis of S_1. If S_R is reciprocal to S_2, then:

$$\xi_2 \circ \xi_R = (h_2 + h_R)\cos\theta = 0.$$

Hence, S_R must always intersect S_1 and either have the negative pitch of S_2, or it can have any pitch if it is orthogonal to the axis of S_2 (i.e. $\cos\theta = 0$).

Problem 5: (25 Points)

The "trick" to this problem is how to orient the manipulator in its "home" position in order to make the analysis straightforward. See Fig. 3 for one appropriate way to do this. In this case, it was necessary to redefine the positive direction of the third joint axis.

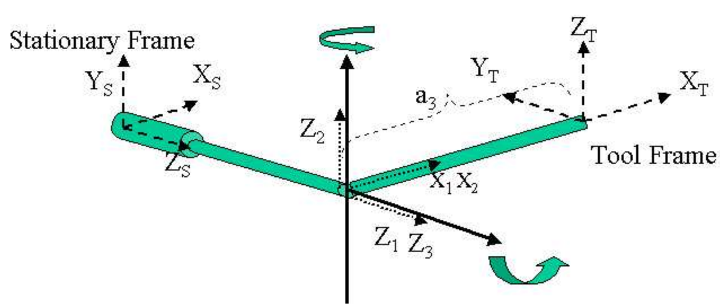

Figure 3: Schematic of PRR Manipulator in "home" position

Part (a) (Denavit-Hartenberg parameters): (5 points) Assuming the definitions shown in Figure 3, the parameters are:

$$\begin{array}{cccc} a_0 = 0 & \alpha_0 = 0 & d_1 = \text{variable} & \theta_1 = 0 \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{variable} \\ a_2 = 0 & \alpha_2 = \frac{\pi}{2} & d_3 = 0 & \theta_3 = \text{variable} \\ a_3 = L & \alpha_3 = -\frac{\pi}{2} & d_4 = 0 & \theta_4 = 0 \end{array} \tag{11}$$

Part (b) (forward kinematics): (5 points) Using the Denavit-hartenberg approach:

$$g_{S1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad g_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_1 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

$$g_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_1 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad g_{3T} = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{13}$$

The total forward kinematics is:

$$g_{ST} = g_{S1}\ g_{12}\ g_{23}\ g_{3T} \ = = \begin{bmatrix} c_2c_3 & -s_2 & c_2s_3 & Lc_2c_3 \\ s_3 & 0 & c_3 & Ls_3 \\ -s_2c_3 & -c_2 & s_2s_3 & d_1 - Ls_2c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{14}$$

where $c_j = \cos\theta_j$, and $s_j = \theta_j$.

Part (c) (Jacobian matrix): (5 points)

The spatial Jacobian has the form:

$$J = \begin{bmatrix} \overline{\xi}_1 & \overline{\xi}_2^{'} & \overline{\xi}_3^{'} \end{bmatrix} \tag{15}$$

where

$$\overline{\xi}_2^{'} = Ad_{e^{d_1\hat{\xi}_1}}\overline{\xi}_2 \tag{16}$$

$$\overline{\xi}_3^{'} = Ad_{e^{d_1\hat{\xi}_1}}Ad_{e^{\theta_2\hat{\xi}_2}}\overline{\xi}_3 \tag{17}$$

Recalling the form of a twist for a prismatic joint, and noting that $d_1 = 0$ in the home position, simple observation of Fig. 3, leads to:

$$\overline{\xi}_1 = \begin{bmatrix} \overline{z}_S \\ \overline{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \overline{\xi}_2 = \begin{bmatrix} \overline{y}_S \\ \overline{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \overline{\xi}_3 = \begin{bmatrix} \overline{0} \\ \overline{z}_S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

Note that:

$$Ad_{e^{d_1\hat{\xi}_1}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -d_1 & 0 \\ 0 & 1 & 0 & d_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad Ad_{e^{\theta_2\hat{\xi}_2}} = \begin{bmatrix} c_2 & 0 & s_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -s_2 & 1 & c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -s_2 & 0 & c_2 \end{bmatrix} \tag{19}$$

Substitution of Eq.s (19) and (18) into Eq.s (16), (17), and (15) yields:

$$J^s_{ST} = \begin{bmatrix} 0 & -d_1 & 0 \\ 0 & 0 & d_1\sin\theta_2 \\ 1 & 0 & 0 \\ 0 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta_2 \end{bmatrix} \tag{20}$$

Part (d) (inverse kinematics): (10 points) Let (x_D, y_D, z_D) denote the desired location of the origin of the tool frame, with respect to th origin of the stationary frame. Let's use the algebraic approach for solving inverse kinematics. Of course, to use this method, it assumes that you got the right equations in part (b) we know that:

$$x_D = Lc_2c_3 \tag{21}$$
$$y_D = Ls_3 \tag{22}$$
$$z_D = d_1 - Ls_2c_3. \tag{23}$$

From Eq. (22) we see that $\sin\theta_3 = \frac{y_D}{L}$. Assuming that $|y_D/L| \leq 1$, we have that

$$\theta_3 = \sin^{-1}\left[\frac{y_D}{L}\right]. \tag{24}$$

Two solutions for θ_3 can be obtained from Eq. (24). The second solution, dubbed $\theta_3^{'}$, is $\theta_3^{'} = \pi - \theta_3$. From Eq. (21) we have two θ_2 solutions for each θ_3 solution:

$$\theta_2 = \cos^{-1}\left[\frac{x_D}{L\cos\theta_3}\right]. \tag{25}$$

The second solution is $\theta_2^{'} = -\theta_2$. Finally, from Eq. (23), we can solve for d_1 for each given (θ_2, θ_3) pair:

$$d_1 = z_D + L\sin\theta_2\cos\theta_3.$$

Problem 6: (10 points)

Let $\overline{p} = [p_x \ p_y]^T$ denote the location of the pole of displacement. Let the displacement be described by the homogeneous matrix g:

$$g = \begin{bmatrix} R & \overline{d} \\ \overline{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & \overline{d} \\ \overline{0}^T & 1 \end{bmatrix}. \tag{26}$$

The homogeneous coordinates for the pole are: $\overline{p}^H == [\overline{p}^T \ 1]^T$, where we know that the pole has the special property that $\overline{p} = (I - R)^{-1}\overline{d}$. A simple calculation shows:

$$g\overline{p}^H = \begin{bmatrix} R & \overline{d} \\ \overline{0}^T & 1 \end{bmatrix} \begin{bmatrix} \overline{p} \\ 1 \end{bmatrix} = \begin{bmatrix} R\overline{p} + \overline{d} \\ 1 \end{bmatrix} = \begin{bmatrix} R(I-R)^{-1}\overline{d} + \overline{d} \\ 1 \end{bmatrix}$$

Let's simplify this term:

$$\begin{aligned} R(I-R)^{-1}\overline{d}+\overline{d} &= [R(I-R)^{-1}+I]\overline{d} \\ &= [R(I-R)^{-1}+I](I-R)(I-R)^{-1}\overline{d} \\ &= [R+I-R](I-R)^{-1}\overline{d} = (I-R)^{-1}\overline{d} \\ &= \overline{p} \end{aligned}$$

Thus,

$$g\overline{p}^H = g\begin{bmatrix}\overline{p}\\ 1\end{bmatrix} = \overline{p}^H \ .$$

To understand the properties of the other two eigenvectors/eigenvalues, note that the determinant of g will always be $+1$. Since the determinant is equal to the product of the eigenvalues, and since one eigenvalue has already been determined to be $+1$, it must be true that the product of the remaining eigenvalues is also $+1$. Hence, the remaining eigenvalues are either both real and reciprocal, or they are both complex conjugates. Note also that the trace of g, which is equal to the sum of its eigenvalues, is always $tr(g) = 1 + 2\cos\theta$, where θ is the amount of rotation specified by g. If λ_1 and λ_2 denote these eigenvalues, then:

$$\begin{aligned} \lambda_1\ \lambda_2 &= 1 \\ \lambda_1 + \lambda_2 &= 2\cos\theta \end{aligned}$$

Hence, the eigenvalues are complex conjugates: $e^{\pm j\theta}$. The associated eigenvectors, which we shall denote by $\overline{e}_1$ and $\overline{e}_2$, will generally be complex and conjugate. As we showed in class for the case of $SO(3)$, we can define two real vectors:

$$\overline{c}_1 = \frac{\overline{e}_1+\overline{e}_2}{2} \qquad \overline{c}_2 = \frac{j(\overline{e}_1-\overline{e}_2)}{2}$$

The action of g on these vectors is equivalent to planar rotation.