ME 115(a): Final Exam (Winter Quarter 2009/2010)

Instructions

- 1. Limit your total time to 5 hours. That is, it is okay to take a break in the middle of the exam if you need to ask a question, or go to dinner, etc.
- 2. You may use any class notes, books, or other written material. You may not discuss this final with other class students or other people except me or the class Teaching Assistants.
- 3. You may use Mathematica, MATLAB, or any software or computational tools to assist you. However, if you find that your solution approach requires a lot of algebra or a lot of computation, then you are probably taking a less than optimal approach.
- 4. The final is due by 5:00 p.m. on the last day of finals.
- 5. The point values are listed for each problem to assist you in allocation of your time.

Problem 1: (20 points)

Consider the three screws, S_1 , S_2 , and S_3 , shown in Figure 1. All three screws are perpendicular to a plane, P, and pass through the corners of an equilateral triangle (whose sides have dimension d). Each of the three screws has zero pitch. Describe the set of all screws which are simultaneously reciptrocal all three screws.



Figure 1: Three Screws

Problem 2: (15 points)

We discovered numerous ways to represent and manipulate spatial displacements. Those crazy kinematicians have yet another variation on the same theme using something called "dual numbers." A dual number, \tilde{a} , takes the form:

$$\tilde{a} = a_r + \epsilon \ a_d$$

where a_r is the "real" part of the dual number and a_d is the "dual" or "pure" part of the dual number. The bases for the dual numbers are 1 and ϵ , and they obey the rules:

$$1 \cdot 1 = 1$$

$$1 \cdot \epsilon = \epsilon \cdot 1 = \epsilon$$

$$\epsilon^2 = 0$$

Dual numbers have many interesting properties, though we will only explore one aspect of their characteristics in this problem.

Part (a): (10 points). We can represent spatial displacements as "dual rotation matrices." That is, if a spatial displacement has the form:

$$g = \begin{bmatrix} R & \overline{p} \\ \overline{0}^T & 1 \end{bmatrix}$$

where $R \in SO(3)$ and $\overline{p} \in \mathbb{R}^3$, then the dual representation of the spatial displacement is:

$$\tilde{g} = R + \epsilon(\hat{p}R)$$

- 1. Show that \tilde{g} is an orthogonal matrix.
- 2. If g_1 and g_2 are spatial displacements, and \tilde{g}_1 and \tilde{g}_2 there dual equivalents, then show that $g_1 g_2$ and $\tilde{g}_1 \tilde{g}_2$ are equivalent.

Hint: in some ways of solving this problem, it might be useful to recall that if $A \in SO(3)$ and $\overline{v} \in \mathbb{R}^3$, then $(A\overline{v}) = A\hat{v}A^T$.

Part (b): (5 Points). We can also use dual numbers to represent twist coordinates. Let $\xi = [\overline{V}, \overline{\omega}]^T$ be a vector of twist coordinates. Its dual representation is $\tilde{\xi} = \overline{\omega} + \epsilon \overline{V}$. Show that

- 1. if g is a spatial displacement, and ξ is a twist, then $Ad_q\xi$ is equivalent to $\tilde{g}\xi$.
- 2. If ξ_1 and ξ_2 are two twists, then the dual part of dual dot product $\tilde{\xi}_1 \cdot \tilde{\xi}_2$ is equivalent to the reciprocal product of ξ_1 and ξ_2 . (Note, the real part of this product is called the "Klein product.").

Problem 3: (25 Points) The first three joints of the "armatron" manipulator (a toy sold by Radio Shack!) are shown in Figure 2



Figure 2: Schematic of Armatron Manipulator Geometry

- **Part (a):** (3 points) Determine the Denavit-Hartenberg parameters.
- **Part (b):** (7 points) Using either the Denavit-hartenberg approach or the product-ofexponentials approach, determine the forward kinematics. That is, relate the coordinates of the origin of the tool frame to the joint angles.
- **Part (c):** (15 points) Solve the inverse kinematics of this manipulator, assuming that the goal is to position the origin of the tool frame. You can use a geometric, algebraic, Paden-Kahan, or other approach.

Problem 4: (17 points)

For the rotation matrix given below

$$R = \begin{bmatrix} 0.833333 & -0.186887 & 0.52022 \\ 0.52022 & 0.583333 & -0.623773 \\ -0.186887 & 0.79044 & 0.583333 \end{bmatrix}$$

Part (a): (8 points) Compute the axis of rotation and angle of rotation.

Part (b): (4 points) Determine the unit quaternion that is equivalent to this rotation.

Part (c): (5 points) What are the z-y-z Euler angles of this rotation?

Problem 5: (15 points)

Planar displacements can be represented as a combination of a translation by vector $\overline{d} = [d_x d_y]^T$ and a rotation by angle θ , which could be also be represented by a 3×3 homogeneous matrix of the form:

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix}$$
(1)

Or, we also noted that every planar displacement was equivalent to a rotation about a "pole."

Part (a): (7 points) Let a body-fixed reference frame attached to a rigid body be initially in coincidence with the origin of a fixed reference observing frame. Let this body undergo planar displacement by rotation of angle ϕ about a pole located at a distance $\bar{p} = [p_x p_y^T \text{ from}$ the origin of the reference frame. Compute the 3×3 homogeneous transformation matrix that describes this displacement (in terms of ϕ , p_x , and p_y).

We can also represent planar displacements using a special type of quaternion algebra knowns as "planar quaternions." The planar quaternion algebra has basis elements $(1, i\epsilon, j\epsilon, k)$ where:

- The product of basis elements i, j, k behave just like the quaternion basis elements.
- $\epsilon^2 = 0.$
- *i*, *j*, and *k*, commute with ϵ . E.g., $i\epsilon \ k = ik \ \epsilon = -j\epsilon$ and $i\epsilon \ j\epsilon = ij\epsilon^2 = 0$.

Recall that in the case of unit quaternions, the quaternion coefficients can be identified with the Euler parameters of a rotation, and thus unit quaternions can be used to represent rotations.

Let a planar quaternion have the form:

$$Z = Z_4 + Z_1 i\epsilon + Z_2 j\epsilon + Z_3 k$$

where $Z_1, Z_2, Z_3, Z_4 \in \mathbb{R}^n$. The coefficients of the planar quaternion can be identified with the planar displacement parameters as follows:

$$Z_4 = \cos\left(\frac{\varphi}{2}\right)$$

$$Z_3 = \sin\left(\frac{\phi}{2}\right)$$

$$Z_2 = -p_x \sin\left(\frac{\phi}{2}\right)$$

$$Z_1 = p_y \sin\left(\frac{\phi}{2}\right)$$
(2)

where $\overline{p} = [p_x \ p_y]^T$ is the pole of the planar displacement.

Part (b): (15 points)

Using your results from part (a), compute ϕ , d_x , and d_y in terms of Z_1 , Z_2 , Z_3 , and Z_4 .

Note: these results are useful because of the following facts, which you need not prove. Let $\overline{v} = (x, y, 1)$ be a planar vector in homogeneous coordinates. This vector can be associated with the "pure" planar quaternion:

$$v = (yi\epsilon - xj\epsilon + k).$$

If Z is a planar quaternion representing a planar displacement with parameters θ , d_x , and d_y , then it can be shown that:

$$\overline{v}' = ZvZ^* = (x\sin\phi + y\cos\phi + d_y)i\epsilon - (x\cos\phi - y\sin\phi + d_x)j\epsilon + k$$
(3)

and thus this operation is equivalent to:

$$\overline{v}' = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \overline{v} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

Thus, planar quaternions are yet another means to represent planar displacements and coordinate transformations.