**ME 115(a): Final Exam Solutions**
(Winter Quarter 2009/2010)

**Problem 1:** (20 points)

You were asked to describe the set of screws which are reciprocal to the zero pitch screws $S_1$, $S_2$, and $S_3$ in Figure 1.

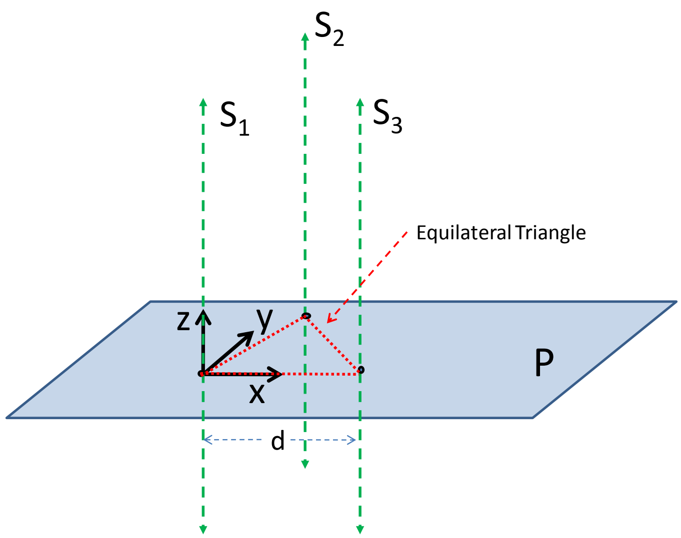


Figure 1: Three Screws

First let's describe the screw coordinates of screws $S_1$, $S_2$, ane $S_3$. Recall that the general screw coordinates for screw $S$ are:

$$S = \begin{bmatrix} \vec{v} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} h\vec{\omega} + \vec{\rho} \times \vec{\omega} \\ \vec{\omega} \end{bmatrix} \tag{1}$$

where $\vec{\omega}$ is a unit vector pointing in the direction of the screw axis, $h$ is the pitch of the screw, and $\vec{\rho}$ is a vector to any point on the screw axis. For screw $S_1$, $h_1 = 0$, $\vec{\rho}_1 = 0$ (since it intersects the origin of the coordinate system) and $\vec{\omega}_1 = \vec{z}$ where $\vec{z} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ is the unit vector $z$-axis. Thus,

$$S_1 = \begin{bmatrix} \vec{v}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} h_1\vec{\omega}_1 + \vec{\rho}_1 \times \vec{\omega}_1 \\ \vec{\omega}_1 \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{z} \end{bmatrix} \tag{2}$$

Similarly, for screws $S_2$ and $S_3$, the screw axes are collinear with the $z$-axis ($\vec{\omega}_2 = \vec{\omega}_3 = \vec{z}$).

Points on the screw axes are respectively: $\vec{\rho}_2 = d(\cos(60^o)\vec{x} + \sin(60^o)\vec{y})$, and $\vec{\rho}_3 = d\vec{x}$.

$$\begin{aligned} S_2 &= \begin{bmatrix} \vec{v}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} h_2\vec{\omega}_2 + \vec{\rho}_2 \times \vec{\omega}_2 \\ \vec{\omega}_2 \end{bmatrix} = \begin{bmatrix} d(\cos(60^o)\vec{x} + \sin(60^o)\vec{y}) \times \vec{z} \\ \vec{z} \end{bmatrix} \\ &= \begin{bmatrix} d(\sin(60^o)\vec{x} - \cos(60^o)\vec{y}) \\ \vec{z} \end{bmatrix} \end{aligned} \quad (3)$$

$$S_3 = \begin{bmatrix} \vec{v}_3 \\ \vec{\omega}_3 \end{bmatrix} = \begin{bmatrix} h_3\vec{\omega}_1 + \vec{\rho}_3 \times \vec{\omega}_3 \\ \vec{\omega}_3 \end{bmatrix} = \begin{bmatrix} d\vec{x} \times \vec{z} \\ \vec{z} \end{bmatrix} = \begin{bmatrix} -d\vec{y} \\ \vec{z} \end{bmatrix} \quad (4)$$

Let $S_R$ be a "reciprocal screw" with screw coordinates:

$$S_R = \begin{bmatrix} \vec{v}_R \\ \vec{\omega}_R \end{bmatrix} = \begin{bmatrix} h_R\vec{\omega}_R + \rho_R \times \vec{\omega}_R \\ \vec{\omega}_R \end{bmatrix} . \quad (5)$$

where the values of $h_R$, $\vec{\omega}_R$ and $\vec{\rho}_r$ are yet to be determined. Recall that the reciprocal product between two screws $S_a = \begin{bmatrix} \vec{v}_a \\ \vec{\omega}_a \end{bmatrix}$ and $S_= \begin{bmatrix} \vec{v}_b \\ \vec{\omega}_b \end{bmatrix}$ is $S_a \odot S_b = \vec{v}_a \cdot \vec{\omega}_b + \vec{\omega}_a \cdot \vec{v}_b$.

First apply the constraint that $S_1$ must be reciprocal to $S_R$:

$$S_R \odot S_1 = \vec{v}_R \cdot \vec{z} + \vec{\omega}_R \cdot \vec{0} = \vec{v}_R \cdot \vec{z} = 0 . \quad (6)$$

Next apply the contraint that $S_2$ must be reciprocal to $S_R$:

$$S_R \odot S_2 = \vec{\omega}_2 \cdot \vec{v}_R + \vec{v}_2 \cdot \vec{\omega}_R = \vec{z} \cdot \vec{v}_R + \vec{v}_2 \cdot \vec{\omega}_R = \vec{v}_2 \cdot \vec{\omega}_R = 0 , \quad (7)$$

where we have used the fact that $\vec{v}_R \cdot \vec{z} = 0$ from the first constraint in order to obtain the result. Similarly, apply the contraint that $S_3$ must be reciprocal to $S_R$:

$$S_R \odot S_3 = \vec{\omega}_R \cdot \vec{v}_3 + \vec{v}_R \cdot \vec{\omega}_3 = \vec{\omega}_R \cdot \vec{v}_3 = 0 \quad (8)$$

where we again have used the fact that $\vec{v}_R \cdot \vec{\omega}_3 = \vec{v}_R \cdot \vec{z} = 0$ from the first constraint. Thus, we know that the screw axis of the reciprocal screw must be orthogonal to both $\vec{v}_2$ and $\vec{v}_3$:

$$\vec{v}_2 = (d\sin(60^o))\vec{x} - (d\cos(60^o))\vec{y} \qquad \vec{v}_3 = -d\vec{y} . \quad (9)$$

Therefore, $\vec{\omega}_R$ must be orthogonal to both the $x$ and $y$ axes, which means that $\vec{\omega}_R = \vec{z}$. Hence all screws reciprocal to $S_1$, $S_2$, and $S_3$ must have axes that are parallel with all three screw axes. Now substitute this result back into the first constraint:

$$S_1 \odot S_R = 0 \quad \Rightarrow \quad \vec{z} \cdot \vec{v}_R = \vec{z} \cdot (h_R\vec{\omega}_R + \vec{\rho}_R \times \vec{\omega}_r) \quad (10)$$

$$= \vec{z} \cdot (h_R\vec{z} + \vec{\rho}_R \times \vec{z}) \quad (11)$$

$$= h_R = 0 \quad (12)$$

Hence, all of the reciprocal screws must be zero pitch screws which are parallel to $S_1$, $S_2$, and $S_3$, and can intersect the plane $P$ at any point. Let's interpret this result in terms of the *virtual work principle*. Let $S_1$, $S_2$, and $S_3$ be motion screws (which correspond to pure rotations about the vertical axes). Then the reciprocal screws must be interpreted in terms of wrenches. The zero pitch reciprocal wrenches then correspond to pure forces (a zero pitch wrench) parallel to the screw axes. Conversely, if $S_1$, $S_2$, and $S_3$ correspond to forces, then the reciprocal screws correspond to rotations about parallel axes.

**Problem 2:** (15 Points)

This problem asked you to look at the represention and manipulation of spatial displacements using the concept of "dual numbers." A dual number, $\tilde{a}$, takes the form:

$$\tilde{a} = a_r + \epsilon\ a_d$$

where $a_r$ is the "real" part of the dual number and $a_d$ is the "dual" or "pure" part of the dual number. The bases for the dual numbers are 1 and $\epsilon$, and they obey the rules:

$$\begin{aligned} 1 \cdot 1 &= 1 \\ 1 \cdot \epsilon &= \epsilon \cdot 1 = \epsilon \\ \epsilon^2 &= 0\ . \end{aligned}$$

**Part (a):** (10 points)

1. $\tilde{g}$ will be orthogonal if $\tilde{g}^T\tilde{g} = I$.

$$\begin{aligned} \tilde{g}^T\tilde{g} &= [R + \epsilon(\hat{p}R)]^T[R + \epsilon(\hat{p}R)] = [R^T - \epsilon R^T\hat{p}][R + \epsilon\hat{p}R] \\ &= R^TR + \epsilon R^T\hat{p}R - \epsilon R^T\hat{p}R - \epsilon^2 R^T\hat{p}^2R \\ &= I + \epsilon R^T(\hat{p} - \hat{p})R = I \end{aligned}$$

2. Let $\tilde{g}_1 = [R_1 + \epsilon\hat{p}_1R_1]$ and $\tilde{g}_2 = [R_2 + \epsilon\hat{p}_2R_2]$. Then:

$$\begin{aligned} \tilde{g}_3 = \tilde{g}_1\tilde{g}_2 &= [R_1 + \epsilon\hat{p}_1R_1][R_2 + \epsilon\hat{p}_2R_2] \\ &= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2) + \epsilon^2(\hat{p}_1R_1\hat{p}_2R_2) \\ &= R_1R_2 + \epsilon(R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2) \end{aligned}$$

Note that $g_3 = g_1g_2$ is given by:

$$g_3 = \begin{bmatrix} R_1R_2 & R_1\vec{p}_2 - \vec{p}_1 \\ \vec{0}^T & 1 \end{bmatrix}$$

Hence, $\tilde{g}_3 = R_1R_2 + \epsilon((\widehat{R_1\vec{p}_2 - \vec{p}_1})R_1R_2)$. Using the "hint" given in the instructions, note that:

$$\begin{aligned} ((\widehat{R_1\vec{p}_2 - \vec{p}_1})R_1R_2) &= (\widehat{R_1\vec{p}_2} - \hat{p}_1)R_1R_2 \\ &= (R_1\hat{p}_2R_1^T - \hat{p}_1)R_1R_2 \\ &= R_1\hat{p}_2R_2 + \hat{p}_1R_1R_2 \end{aligned}$$

Therefore, the two calculations are equivalent.

**Part(b):** (5 points)

1. Let a transformation $g$ consists of rotation $R$ and displacement $\vec{p}$. Let $\vec{\xi} = \begin{bmatrix}\vec{V}^T & \vec{\omega}^T\end{bmatrix}^T$ be a twist vector. Then:

$$Ad_g\xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{bmatrix} \vec{V} \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} R\vec{V} + \hat{p}R\vec{\omega} \\ R\vec{\omega} \end{bmatrix}$$

   The "dual" version of this vector is $R\vec{\omega} + \epsilon(R\vec{V} + \hat{p}R\vec{\omega})$. But, $\tilde{g} = R + \epsilon(\hat{p}R)$ and $\tilde{\xi} = \vec{\omega} + \epsilon\vec{V}$. Hence:

$$\begin{aligned} \tilde{g}\tilde{\xi} &= (R + \epsilon(\hat{p}R))(\vec{\omega} + \epsilon\vec{V}) && (13) \\ &= R\vec{\omega} + \epsilon(R\vec{V} + \hat{p}R\vec{\omega}) + \epsilon^2(\hat{p}R\vec{V}) && (14) \\ &= R\vec{\omega} + \epsilon(R\vec{V} + \hat{p}R\vec{\omega}) && (15) \end{aligned}$$

   Thus, the two calculations are equivalent.

2. Let $\xi_1 = [\vec{V}_1^T\ \vec{\omega}_1^T]^T$ and $\xi_2 = [\vec{V}_2^T\ \vec{\omega}_2^T]^T$. Then:

$$\begin{aligned} \tilde{\xi}_1 \cdot \tilde{\xi}_2 &= (\vec{\omega}_1 + \epsilon\vec{V}_1) \cdot (\vec{\omega}_2 + \epsilon\vec{V}_2) = \vec{\omega}_1 \cdot \vec{\omega}_2 + \epsilon(\vec{V}_1 \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot \vec{V}_2) + \epsilon^2(\vec{V}_1 \cdot \vec{V}_2) \\ &= \vec{\omega}_1 \cdot \vec{\omega}_2 + \epsilon(\vec{V}_1 \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot \vec{V}_2) \end{aligned}$$

   The dual part of this, $\vec{V}_1 \cdot \vec{\omega}_2 + \vec{\omega}_1 \cdot \vec{V}_2$, is the reciprocal product of $\xi_1$ and $\xi_2$.

**Problem 3:** (25 Points)

This problem analyzed the kinematics of the "armatron" manipulator toy of Figure 2.

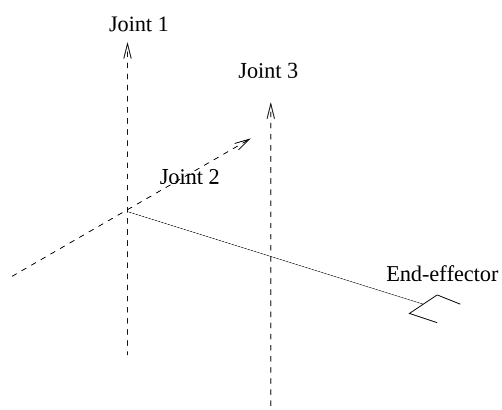


Figure 2: Schematic of Armatron Manipulator Geometry

**Part (a):** (3 points) Determine Denavit-Hartenberg parameters.

Assume that the origin of the stationary frame is placed at the intersection of joints axes 1 and 2, with the z-axis collinear with joint axis 1, and the y-axis collinear with joint axis 2 (when it is in the "zero" position). If the positive directions of the joint axes are chosen as in the figure, then

$$\begin{array}{llll} a_0 = 0 & \alpha_0 = 0 & d_1 = 0 & \theta_1 = \text{variable} \\ a_1 = 0 & \alpha_1 = -\frac{\pi}{2} & d_2 = 0 & \theta_2 = \text{variable} \\ a_1 = l_1 & \alpha_2 = \frac{\pi}{2} & d_3 = 0 & \theta_2 = \text{variable} \\ a_3 = l_2 & \alpha_3 = 0 & & \end{array} \tag{16}$$

**Part (b):** (7 points) Determine the forward kinematics. Using the Denavit-hartenberg approach:

$$g_{S1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad g_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_1 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

$$g_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ \sin\theta_1 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad g_{3T} = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

The total forward kinematics is:

$$g_{ST} = g_{S1}\ g_{12}\ g_{23}\ g_{3T} \ = \begin{bmatrix} R_{ST} & \vec{p}_{ST} \\ \vec{0}^T & 1 \end{bmatrix} \tag{19}$$

where:

$$\vec{p}_{ST} = \begin{bmatrix} l_1 c_1 c_2 + l_2(c_1 c_2 c_3 - s_1 s_3) \\ l_1 s_1 c_2 + l_2(s_1 c_2 c_3 + c_1 s_3) \\ -l_1 s_2 - l_2 s_2 c_3 \end{bmatrix} \tag{20}$$

and the short hand notation $c_1 = \cos\theta_1$, $s_2 = \sin\theta_2$, etc. is use for simplicity.

**Part (c):** (15 points) Inverse kinematics. The goal is to solve the equations:

$$\vec{p}_{ST} = \begin{bmatrix} l_1 c_1 c_2 + l_2(c_1 c_2 c_3 - s_1 s_3) \\ l_1 s_1 c_2 + l_2(s_1 c_2 c_3 + c_1 s_3) \\ -l_1 s_2 - l_2 s_2 c_3 \end{bmatrix} = \begin{bmatrix} x_T \\ y_T \\ z_T \end{bmatrix} \tag{21}$$

where $(x_T, y_T, z_T)$ are the Cartesian coordinates of the origin of the tool frame (with respect to the origin of the Stationary frame described above).

Note that:

$$R^2 = x_T^2 + y_T^2 + z_T^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos\theta_3. \tag{22}$$

Two solutions for $\theta_3$ can be obtained from Eq. (22). In analogy to the elbow manipulator, one can think of these two solutions as a "right elblow" and "left elbow" solution.

The third component of the forward kinematics equations is $z_T = -\sin\theta_2(l_1 + l_2\cos\theta_3)$. Substituting for the now known value of $\cos\theta_3$ from Eq. (22) yields:

$$z_T = -\sin\theta_2(l_1 + l_2\cos\theta_3) = \sin\theta_2\left(l_1 + \frac{R^2 - l_1^2 - l_2^2}{2l_1}\right). \tag{23}$$

This can be solved directly for an expression in $\sin\theta_2$, which can then be inverted to find two solutions for each $\theta_3$ solution:

$$\theta_2 = \sin^{-1}\left[\frac{2l_1 z_T}{R^2 + l_1^2 - l_2^2}\right] \tag{24}$$

Now that $\theta_2$ and $\theta_3$ are known, the $x$- and $y$- components of Equation (21) can be solved to find the unique $\theta_1$ for each $(\theta_2, \theta_3)$ pair:

$$x_T = c_1(l_1c_2 + l_2c_2c_3) - s_1(l_2s_3) = Ac_1 - Bs_1 \tag{25}$$

$$y_T = s_1(l_1c_2 + l_2c_2c_3) + c_1(l_2s_3) = As_1 + Bc_1 \tag{26}$$

where $A = l_1c_2 + l_2c_2c_3$ and $B = l_2s_3$. These two equations (25) and (26) can be solved for the unknowns $s_1$ and $c_1$

$$s_1 = \frac{Ay_T - Bx_T}{A^2 + B^2} \qquad c_1 = \frac{Ax_T + By_T}{A^2 + B^2}\ . \tag{27}$$

From Equation (27) it is clear that the unique value of $\theta_1$ associated with each one of the four different $(\theta_1, \theta_2)$ pairs is:

$$\theta_1 = Atan2[(Ay_T - Bx_T), (Ax_T + By_T)]\ . \tag{28}$$

**Problem 4:** (17 points). This problem concerned various rotation representations related to the matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 0.833333 & -0.186887 & 0.52022 \\ 0.52022 & 0.583333 & -0.623773 \\ -0.186887 & 0.79044 & 0.583333 \end{bmatrix}.$$

**Part (a):** (8 points). Compute the axis of rotation and the angle of rotation. The angle of rotation can be computed from the formula:

$$\cos\phi = \frac{r_{11} + r_{22} + r_{33} - 1}{2} = \frac{0.833333 + 0.583333 + 0.583333 - 1}{2} = 0.5$$

which yields solutions $\phi_1 = \cos^{-1}(0.5) = 60^o$ and $\phi_2 = -\phi_1 = -60^o$. The $x$, $y$, and $z$ components of the unit vector axis of rotation can be found as:

$$\omega_x = \frac{r_{32} - r_{23}}{2\sin(60^o)} = \frac{0.79044 + 0.623773}{2 * 0.866025} = 0.816496 \tag{29}$$

$$\omega_y = \frac{r_{13} - r_{31}}{2\sin(60^o)} = \frac{0.52022 + 0.0.186887}{2 * 0.866025} = 0.408248 \tag{30}$$

$$\omega_z = \frac{r_{21} - r_{12}}{2\sin(60^o)} = \frac{0.52022 + 0.186887}{2 * 0.866025} = 0.408248 \tag{31}$$

**Part (b):** (4 points). The unit quarternion equivalent to this rotation is given by:

$$\begin{aligned} q &= (a,b,c,d) = (\cos(\frac{\phi}{2}), \omega_x \sin(\frac{\phi}{2}), \omega_y \sin(\frac{\phi}{2}), \omega_z \sin(\frac{\phi}{2})) \\ &= (0.866025, 0.408248, 0.204124, 0.204124) \end{aligned} \tag{32}$$

**Part (c):** (5 points). The z-y-z Euler were calculated in the class notes. If the successive angles are denoted $\psi$, $\phi$, and $\gamma$, then:

$$\cos\phi = r_{33} \Rightarrow \phi = \cos^{-1}(0.58333) = 54.3147^o \tag{33}$$

$$\gamma = Atan2[\frac{r_{32}}{\sin\phi}, \frac{-r_{31}}{\sin\phi}] = Atan2[0.973169, 0.23009] = 76.6967^o \tag{34}$$

$$\psi = Atan2[\frac{r_{23}}{\sin\phi}, \frac{r_{13}}{\sin\phi}] = Atan2[-0.767973, 0.640481] = -50.1723^o \tag{35}$$

**Problem 5:** (15 points). This problem concerned the concepted of "planar quaternions."

**Part (a):** The homogeneous displacement is most easily computed via a similarity transform. That is, let $g_R$ denote the displacement which is pure rotation about the pole:

$$g_R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $g_D$ denote the displacment from the reference frame to a frame whose origin is at the pole, and with basis vectors parallel to the reference frame basis vectors:

$$g_D = \begin{bmatrix} 1 & 0 & p_x \\ 0 & 1 & p_y \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the displacement of the rigid body is:

$$g = g_D g_R g_D^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & p_x(1-\cos\theta) + p_y \sin\theta \\ \sin\theta & \cos\theta & -p_x \sin\theta + p_y(1-\cos\theta) \\ 0 & 0 & 1 \end{bmatrix}$$

In other words, the displacement vector is $\vec{d} = (\mathbf{I} - \mathbf{A})\vec{p}$.

**Part (b):** Noting that:

$$\begin{aligned} \cos\theta &= \cos^2\tfrac{\theta}{2} - \sin^2\tfrac{\theta}{2} = Z_4^2 - Z_3^2 \\ \sin\theta &= 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = 2Z_3Z_4 \end{aligned}$$

then:

$$
\begin{aligned}
d_x &= p_x(1-\cos\theta) + p_y \sin\theta \\
&= 2[p_x \sin^2\tfrac{\theta}{2} + p_y \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}] \\
&= 2[-Z_2 Z_3 + Z_1 Z_4] \\
d_y &= -p_x \sin\theta + p_y(1-\cos\theta) \\
&= 2[-p_x \sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} + p_y \sin^2\tfrac{\theta}{2}] \\
&= 2[Z_2 Z_4 + Z_1 Z_3] \\
\theta &= 2Atan2[Z_3, Z_4]
\end{aligned}
$$