ME 115(a): Final Exam

Instructions

1. Limit your total time to 5 hours. That is, it is okay to take a break in the middle of the exam if you need to ask me a question, or go to dinner, etc.
2. You may use any class notes, books, or other written material. You may not discuss this final with other students or other people except me or the T.A.s.
3. You may use mathematica or any software or computational tools to assist you. However, if you find that solution approach requires a lot of algebra or a lot of computation, then you are probably taking a less than optimal approach.
4. The final is due by 5:00 p.m. on the last day of finals.
5. The point values are listed for each problem to assist you in allocation of your time.

Problem 1: (30 Points)

Figure 1 shows a schematic of an 3-jointed "cylindrical" robot manipulator. This manipulator consists of two revolute joints (joints #1 and #2) and one prismatic joint (the third joint). All three joint axes are vertical and parallel to each other.

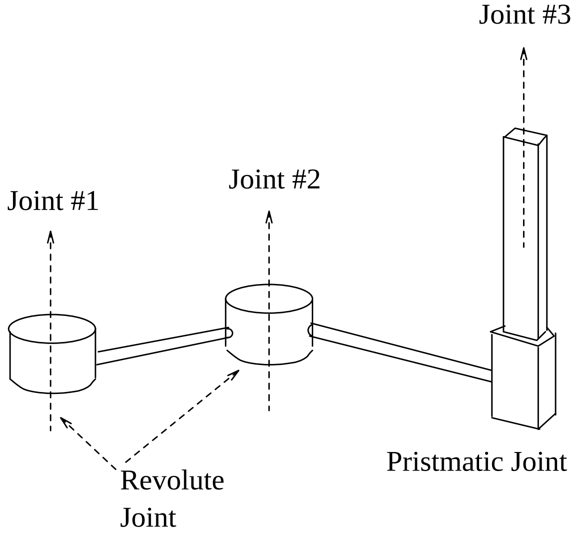

Figure 1: Schematic of a "Cylindrical Manipulator"

- (5 points) Derive the Denavit-Hartenberg parameters.
- (10 points) Derive the forward kinematic equations of this robot. You may use the Denavit-Hartenberg approach, or the product of exponentials approach.
- (15 points) Derive the inverse kinematic solution, assuming that the goal is to position the tool frame origin at some desired position, (x_T, y_T, z_T).

Problem 2: (15 Points)

We discovered numerous ways to represent and manipulate spatial displacements. Those crazy kinematicians have yet another variation on the same theme using something called "dual numbers." A dual number, $\tilde{a}$, takes the form:

$$\tilde{a} = a_r + \epsilon\ a_d$$

where a_r is the "real" part of the dual number and a_d is the "dual" or "pure" part of the dual number. The bases for the dual numbers are 1 and ϵ, and they obey the rules:

$$\begin{aligned} 1 \cdot 1 &= 1 \\ 1 \cdot \epsilon &= \epsilon \cdot 1 = \epsilon \\ \epsilon^2 &= 0 \end{aligned}$$

Dual numbers have many interesting properties, though we will only explore one aspect of their characteristics in this problem.

Part (a): (10 points). We can represent spatial displacments as "dual rotation matrices." That is, if a spatial displacement has the form:

$$g = \begin{bmatrix} R & \overline{p} \\ \overline{0}^T & 1 \end{bmatrix}$$

where $R \in SO(3)$ and $\overline{p} \in \mathbb{R}^3$, then the dual representation of the spatial displacement is:

$$\tilde{g} = R + \epsilon(\hat{p}R)$$

1. Show that $\tilde{g}$ is an orthogonal matrix.
2. If g_1 and g_2 are spatial displacements, and $\tilde{g}_1$ and $\tilde{g}_2$ there dual equivalents, then show that $g_1\ g_2$ and $\tilde{g}_1\ \tilde{g}_2$ are equivalent.

Hint: in some ways of solving this problem, it might be useful to recall that if $A \in SO(3)$ and $\overline{v} \in \mathbb{R}^3$, then $(\widehat{A\overline{v}}) = A\hat{v}A^T)$.

Part (b): (5 Points). We can also use dual numbers to represent twist coordinates. Let $\xi = [\overline{V}, \overline{\omega}]^T$ be a vector twist coordinates. Its dual representation is $\tilde{\xi} = \overline{\omega} + \epsilon\overline{V}$. Show that

1. if g is a spatial displacement, and ξ is a twist, then $Ad_g\xi$ is equivalent to $\tilde{g}\tilde{\xi}$.
2. If ξ_1 and ξ_2 are two twists, then the dual part of dual dot product $\tilde{\xi}_1 \cdot \tilde{\xi}_2$ is equivalent to the reciprocal product of ξ_1 and ξ_2. (Note, the real part of this product is called the "Klein product.").

Problem 3: (20 Points)

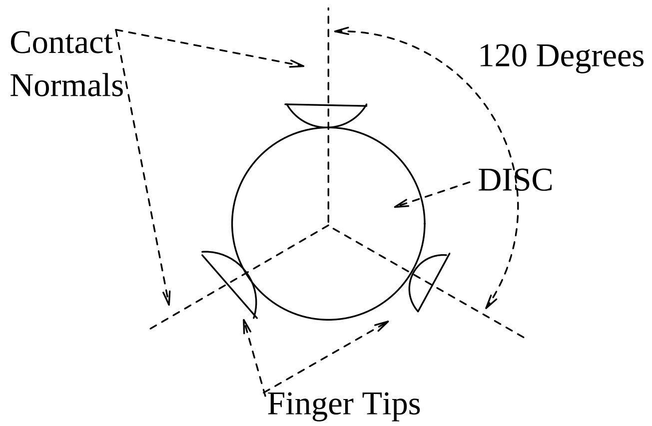

Figure 2: 3-fingered grasp of a disc

Next quarter we will extensively study the problem of grasping–i.e., how one can grab an object with fingers in such as way as to prevent the grasped object from slipping out of the grasp. Consider the picture below in which a planar disc is touched by 3 "planar" fingers. Assume that each finger touches the disc with *frictionless* point contact. Also assume that each finger can apply any possible force to the object.

Question: Is the disc immobilized? That is, are there any free motions of the disc that can not be prevented by the fingers? In addition to an intuitive discussion of this question, you must back up your answer with some analysis.

Problem 4: (20 Points)

Consider the two screws, S_1 and S_2, shown in Figure 3. The screw axis of S_1 is perpendicular to the plane, P, and has zero pitch: $h_1 = 0$. Similarly, the screw axis of S_2 is also perpendicular to the plane. It intersects the plane a distance a from the intersection of S_1, and its pitch is also zero: $h_2 = 0$. Describe the set of all screws whose axes lie in P and that are reciprocal to both S_1 and S_2.

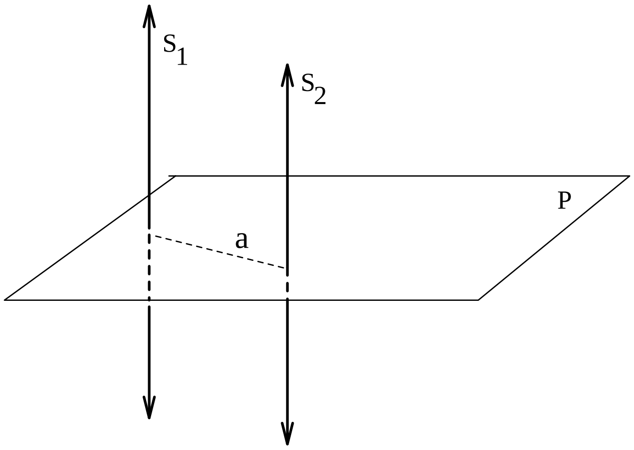

Figure 3: Two Screws

Problem 5: (15 points)

Planar displacements can be represented as a combination of a translation by vector $\overline{d} = [d_x\ d_y]^T$ and a rotation by angle θ, which could be also be represented by a 3×3 homogeneous matrix of the form:

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & d_x \\ \sin\theta & \cos\theta & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

Or, we also noted that every planar displacement was equivalent to a rotation about a "pole."

Part (a): (7 points) Let a reference frame attached to a rigid body be initially in coincidence with the origin of a reference frame. Let this body undergo planar displacement by rotation of angle ϕ about a pole located at a distance $\overline{p} = [p_x\ p_y^T$ from the origin of the reference frame. Compute the 3×3 homogeneous transformation matrix that describes this displacment (in terms of ϕ, p_x, and p_y).

We can also represent planar displacements using a special type of quaternion algebra knowns as "planar quaternions." The planar quaternion algebra has basis elements (1, $i\epsilon$, $j\epsilon$, k) where:

- The product of basis elements i, j, k behave just like the quaternion basis elements.
- $\epsilon^2 = 0$.
- i, j, and k, commute with ϵ. E.g., $i\epsilon\ k = ik\ \epsilon = -j\epsilon$ and $i\epsilon\ j\epsilon = ij\epsilon^2 = 0$.

Recall that in the case of unit quaternions, the quaternion coefficients can be identified with the Euler parameters of a rotation, and thus unit quaternions can be used to represent rotations.

Let a planar quaternion have the form:

$$Z = Z_4 + Z_1 i\epsilon + Z_2 j\epsilon + Z_3 k$$

where Z_1, Z_2, Z_3, $Z_4 \in \mathbb{R}^n$. The coefficients of the planar quaternion can be identified with the planar displacement parameters as follows:

$$\begin{aligned} Z_4 &= \cos(\tfrac{\phi}{2}) \\ Z_3 &= \sin(\tfrac{\phi}{2}) \\ Z_2 &= -p_x \sin(\tfrac{\phi}{2}) \\ Z_1 &= p_y \sin(\tfrac{\phi}{2}) \end{aligned} \tag{1}$$

where $\overline{p} = [p_x\ p_y]^T$ is the pole of the planar displacement.

Part (b): (15 points)

Using your results from part (a), compute ϕ, d_x, and d_y in terms of Z_1, Z_2, Z_3, and Z_4.

Note: these results are useful because of the following facts, *which you need not prove.* Let $\overline{v} = (x, y, 1)$ be a planar vector in homogeneous coordinates. This vector can be associated with the "pure" planar quaternion:

$$v = (yi\epsilon\ -\ xj\epsilon\ +\ k).$$

If Z is a planar quaternion representing a planar displacement with parameters θ, d_x, and d_y, then it can be shown that:

$$\overline{v}' = ZvZ^* = (x\sin\phi + y\cos\phi + d_y)i\epsilon - (x\cos\phi - y\sin\phi + d_x)j\epsilon + k \tag{2}$$

and thus this operation is equivalent to:

$$\overline{v}' = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \overline{v} + \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

Thus, planar quaternions are yet another means to represent planar displacements and coordinate transformations.