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2/9/14

ME 115a - Lecture Notes

Agenda

- Office Hour Scheduling
- Recap of Recent Stuff
- New Stuff
  - Twist Coordinates
  - Rigid Body Velocity
    - Rotational
    - Translational
    - as "Twists"
  - "Types" of velocity
    - Body Velocity
    - Spatial Velocity

Recap

→ A force french fellow named Charles showed that any spatial displacement of a Rigid body was equivalent to a 'screw motion', so called because of the similarity to the

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motivating a screw

Parameters: $(\vec{\omega}, \phi, d'', \vec{p}_\perp)$

axis, rotation angle, translation amount, axis base position

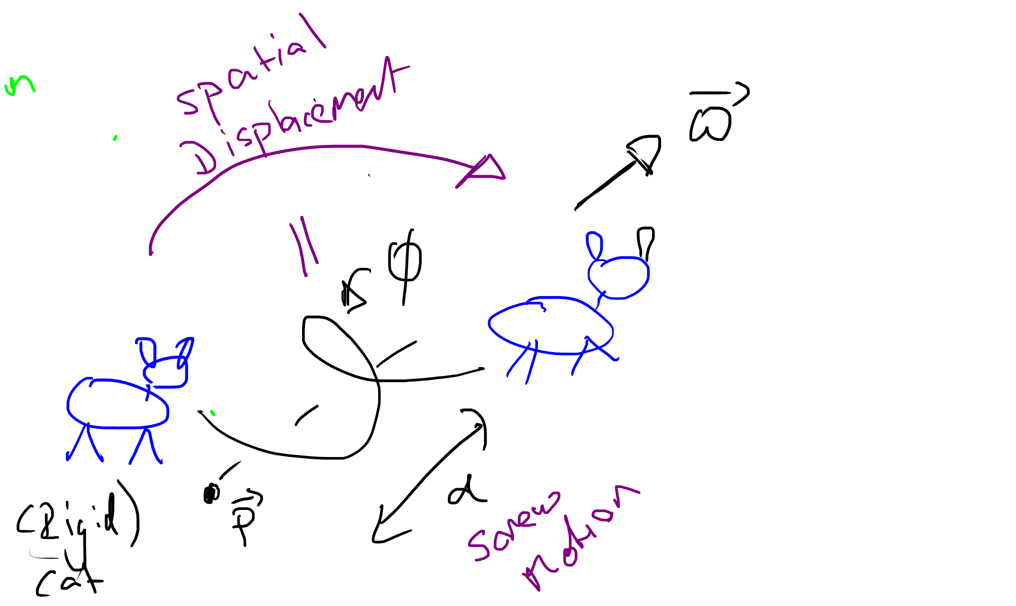

Fig 1. A screwy rigid cat

Conversion into homogenous transformation:

$$g = \left[\begin{array}{c|c} e^{\hat{\omega}\phi} & d''\vec{\omega} + (I - A)\vec{p}_\perp \\ \hline \vec{0} & 1 \end{array}\right]$$

New Stuff

Twist Coords. for Spatial Displacements

Let $g = \begin{bmatrix} \underset{\text{rotation}}{R} & \underset{\text{translation}}{d} \\ \vec{0}^T & 1 \end{bmatrix}$ be a homogenous transformation. Suppose that for some vector $\xi = \begin{bmatrix} \vec{v} \\ \vec{\omega} \end{bmatrix} \in \mathbb{R}^6$, & some $\phi \in \mathbb{R}$, $g = \exp(\hat{\xi}\phi)$

Recall/note that $\hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & \vec{v} \\ \hline \vec{0} & 1 \end{array}\right]$

Calculating $\exp(\phi\hat{\xi})$, we get $\exp(\phi\hat{\xi}) = \left[\begin{array}{c|c} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})(\vec{\omega}\times\vec{v}) + \vec{\omega}\vec{\omega}^T v\,\phi \\ \hline \vec{0}^T & 1 \end{array}\right]$

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equating to g in terms of screw parameters, we get

$$\left[\begin{array}{c|c} e^{\hat{\omega}\phi} & d''\vec{\omega} + (I - e^{\hat{\omega}\phi})\rho_\perp \\ \hline \vec{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} e^{\phi\omega} & (I - e^{\phi\hat{\omega}})(\vec{\omega} \times \vec{v}) + \vec{\omega}\vec{\omega}^T v\,\phi \\ \hline \vec{0}^T & 1 \end{array}\right]$$

If you guessed that $\vec{v}$ takes the form

$$\vec{v} = \vec{\rho} \times \vec{\omega} + h\vec{\omega}, \quad (\vec{\omega} = \vec{\omega})$$

you (would be weird, and you would) find that equality above holds for $\vec{\rho} = \vec{\rho}_\perp + \lambda\vec{\omega}$

$\lambda \in \mathbb{R}$, arbitrary.

SANITY CHECK

Why's the $\underbrace{\text{exponential}}$ showing up?

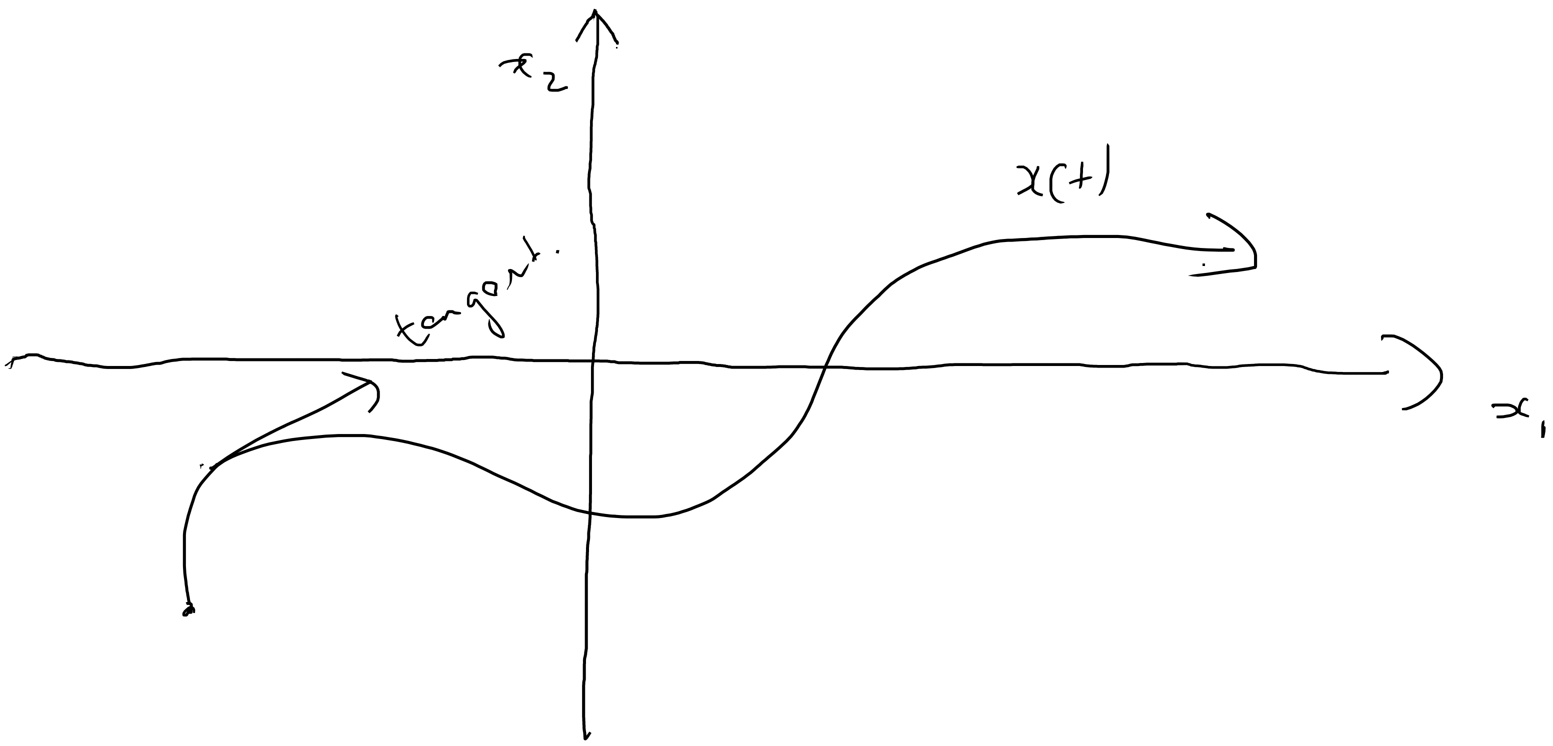
Well... exponentials show up all the time, but in particular, consider

$$\underbrace{\dot{\vec{x}} = A\vec{x}}_{\text{Linear}}$$

constant matrix

solution is $x(t) = e^{At}x_0$.

suppose that $\vec{x}(t)$ is a position in $\mathbb{R}^2$.



The constant 'A' describes tangent vectors! The "derivative" of $\vec{x}$!

now, suppose that $x(t)$ was the path of a rigid body, & included position & orientation as a function of time.
We could think of $\vec{x}$ as a homogenous transformation;
then, $\frac{d}{dt}(x)$ would look like / have something to do with ξ!

"TWIST MATRIX"

And on that note . . .

Rigid Body Velocity

Take my word that all this formal construction for spatial displacements is extremely useful.
But something is missing . . . real things don't undergo non-zero displacements instantaneously!

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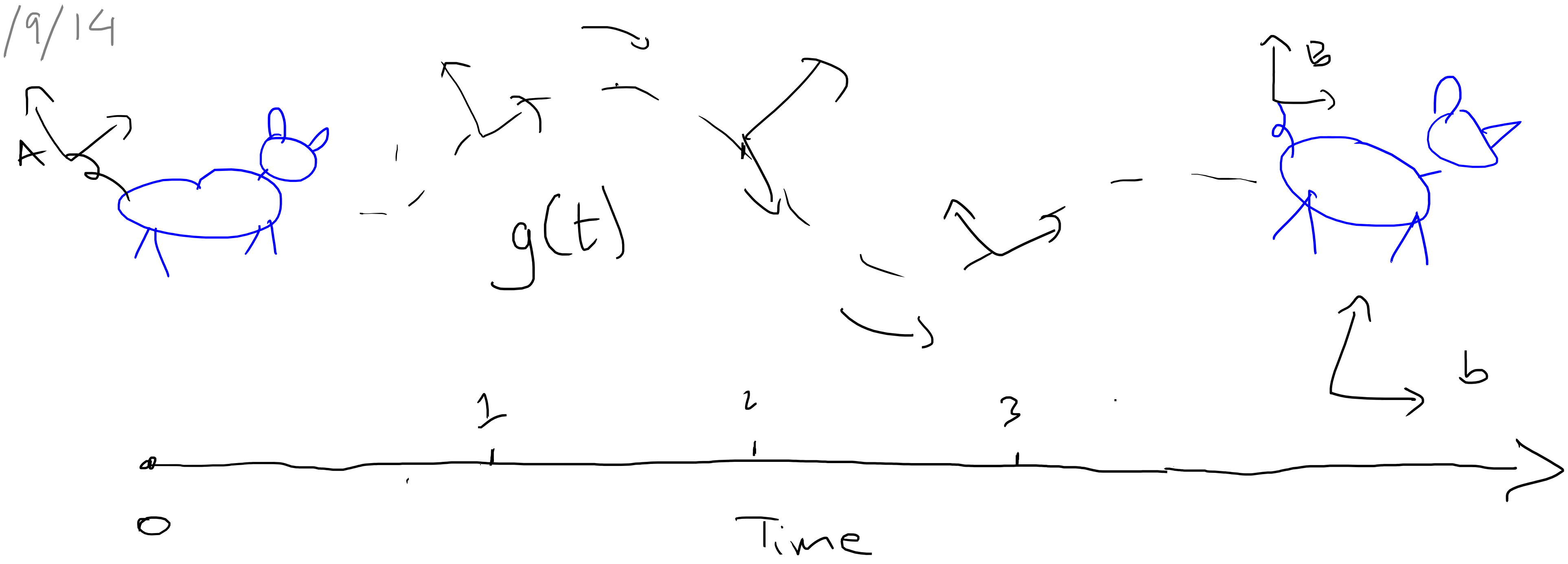
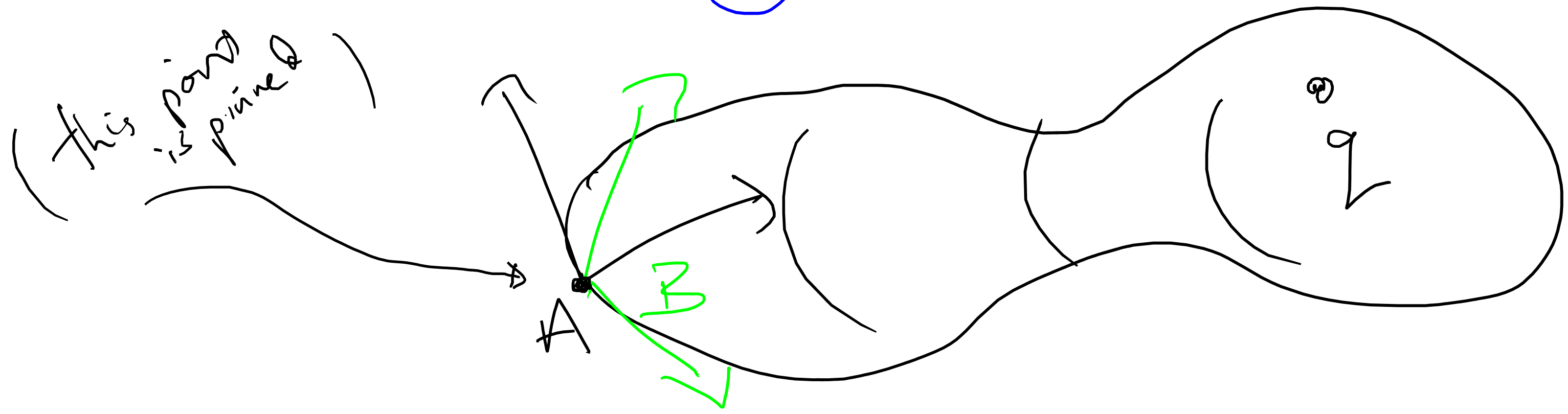


Fig 2. flying (Rigid) cat.

Suppose $g(t) = (R(t), p(t))$. Is there a notion of velocity? Something like $\dot{g}$? $\leftarrow \frac{d}{dt}(g(t))$

Rotational Velocity



Suppose a rigid body is undergoing rotation about a fixed point at the origin of a fixed frame, A. Let B share its origin with A, but rotate along with the body. B is known generally as a "Body frame".

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Let $\underline{\underline{R_{ab}(t)}} : \mathbb{R}_{\geq 0} \to SE(3)$ describe the orientation of frame B with respect to frame A at each time t.

For q_b a point on the body expressed in the B frame,

point q expressed in terms of frame A $\Rightarrow$ $q_a(t) = R_{ab}(t)\, \underbrace{q_b(t)}_{\text{constant}} = R_{ab}(t)\, q_b$

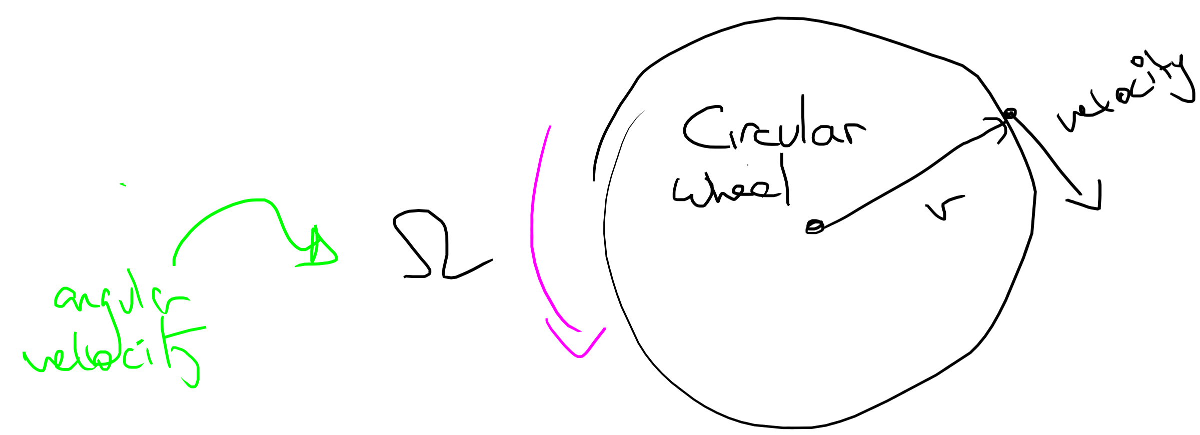
Lets try to work out velocity:

$\Rightarrow$ in spatial frame: $\vec{V}_{q_a} = \frac{d}{dt} q_a(t) = \underline{\dot{R}_{ab}(t)}\, q_b$

map from body coordinate to spatial velocity.

* General note: $[\cdot]_{ab}$ eats things from frame b on the right & spits out things in frame a on the left

I hope you've seen angular velocity in physics/classical mechanics e.g.



velocity at radius r $= \Omega r$.

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Lets see if we can figure out something analogous in the case of 3 dimensional rotational velocity:

Recall : $R_{ab}(t)\, R_{ab}^T(t) = I$ (since $R_{ab} \in SE(3)$)

differentiate using product rule $\Longrightarrow$ $\dot{R}_{ab} R_{ab}^T + R_{ab} \dot{R}_{ab}^T = 0$ ← (zero matrix)

$$\Rightarrow \dot{R}_{ab} R_{ab}^T = - R_{ab} \dot{R}_{ab}^T$$

$\Longrightarrow$ $\dot{R}_{ab} R_{ab}^T$ is skew symmetric!

Lets use this fact:

$$V_{q_a}(t) = \dot{R}_{ab}(t)\, q_b$$

$$= \underbrace{\dot{R}_{ab}(t)\, R_{ab}(t)^T}_{\text{skew sym bro!}} R_{ab}(t)\, q_b$$

Lets Define $\hat{\omega}_{ab}^s(t) = \dot{R}_{ab}(t)\, R_{ab}(t)^T$.

Then,

$$V_{q_a}(t) = \hat{\omega}_{ab}^s(t)\, \underbrace{R_{ab}(t)\, q_b}_{\text{is } q_a!}$$

$$\boxed{\begin{aligned} V_{q_a}(t) &= \hat{\omega}_{ab}^s(t)\, q_a \\ V_{q_a}(t) &= \vec{\omega}_{ab}^s(t) \times q_a \end{aligned}}$$

$\vec{\omega}_{ab}^s$ here is the body's <u>Spatial Angular Velocity</u>

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The quantity $R_{ab}^T \dot{R}_{ab}$ is skew symm by a similar argument to the one above,

Setting $\hat{\omega}_{ab}^b = R_{ab}^T \dot{R}_{ab}$

$$V_{q_b}(t) = R_{ab}^T(t)\, V_{q_a}(t)$$

$$= R_{ab}^T(t)\, \dot{R}_{ab}(t)\, q_b$$

$$\boxed{\begin{aligned} V_{q_b}(t) &= \hat{\omega}_{ab}^b\, q_b \\ &= \vec{\omega}_{ab}^b \times q_b \end{aligned}}$$

$\vec{\omega}_{ab}^b$ is called the body's body angular velocity.

$V_{q_b}(t)$ is the velocity of point q as seen in the B frame

Spatial = (Rotational + Translational) Velocities

It's "easy" to imagine that we might generalize this kind of thing (quantities that capture instantaneous velocity at time t) to arbitrary displacements.

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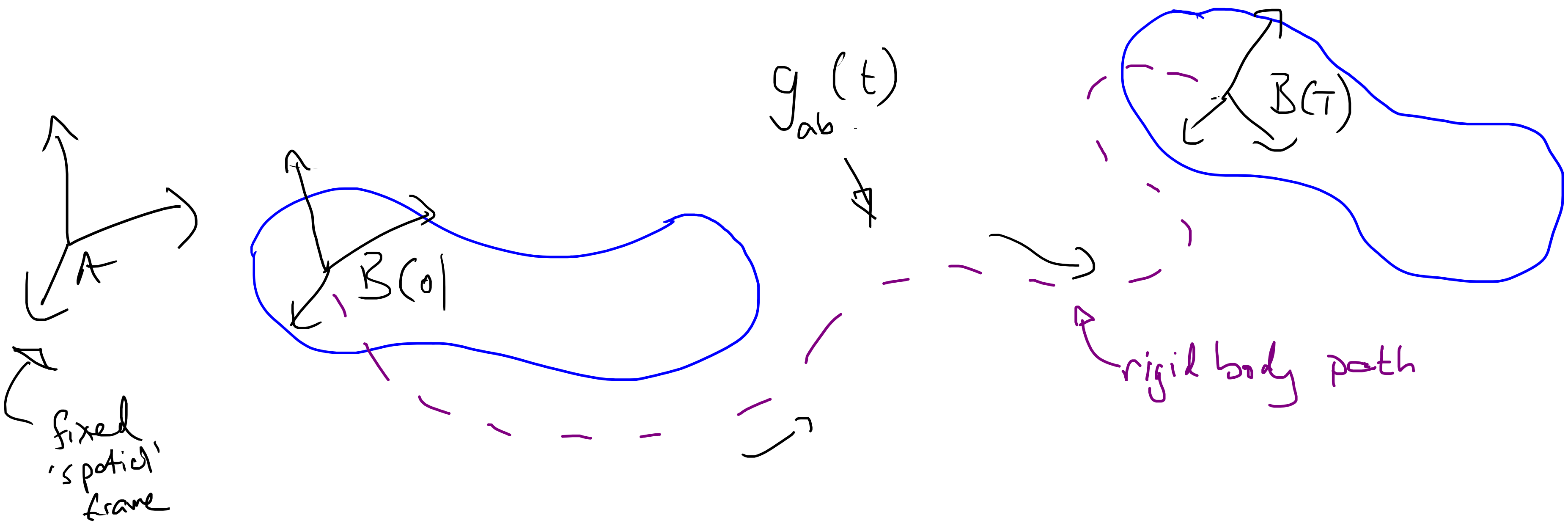
More general picture:



Suppose that

$$g_{ab}(t) = \begin{bmatrix} R_{ab}(t) & d_{ab}(t) \\ 0^T & 1 \end{bmatrix}$$

Let's tinker with derivatives along the same lines as we did for rotations:

$$\dot g_{ab} g_{ab}^{-1} = \underbrace{\begin{bmatrix} \dot R_{ab} & \dot d_{ab} \\ 0^T & 0 \end{bmatrix}}_{\dot g_{ab}} \underbrace{\begin{bmatrix} R_{ab}^T & -R_{ab}^T d_{ab} \\ 0 & 1 \end{bmatrix}}_{g_{ab}^{-1}}$$

$$= \begin{bmatrix} \dot R_{ab} R_{ab}^T & (\dot d_{ab} - R_{ab} R_{ab}^T d_{ab}) \\ 0^T & 0 \end{bmatrix}$$

Look familiar?

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Lets Define

$$\boxed{\hat{V}^s_{ab} = \left[\begin{array}{c|c} \dot{R}_{ab}R^T_{ab} & (\dot{d}_{ab} - \dot{R}_{ab}R^T_{ab}d_{ab}) \\ 0^T & 0 \end{array}\right]}$$

The "Spatial Velocity" of the Rigid Body.

We can 'un-hat' $\hat{V}^s_{ab}$ to get

$$V^s_{ab} = \begin{bmatrix} v_{ab} \\ \omega_{ab} \end{bmatrix} \in se(3) \quad \text{where } v_{ab} = (\dot{d}_{ab} - \dot{R}_{ab}R^T_{ab}d_{ab})$$

$$\hat{\omega}_{ab} = \dot{R}_{ab}R^T_{ab}$$

Remember this?

$\hat{V}^s_{ab}$ is a twist matrix and V^s_{ab} is the corresponding twistvector. The set of twist vectors is isomorphic to twist matrices. We can think of R.B. velocities either way, and call them "twists" to save oxygen.

isomorphism: A bijective map that preserves some structure.

In this case: $V^s_{ab} = \underbrace{(\hat{V}^s_{ab})^\vee}_{\text{our map}} = \begin{bmatrix} v_{ab} \\ \omega_{ab} \end{bmatrix}$.

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The structure that is preserved is the vector space structure of $se(3)$.

How do we interpret spatial velocities?

① $\bar{\omega}^s_{ab}$ = instantaneous axis of rotation

weird Alert! → ② $\bar{v}^s_{ab}$ = velocity of particle coincident to origin of frame A.

A little weird! Mull this over a few times!

We can do the same thing we did for rotational motion to get Body Velocities,

$$\text{Set} \quad \hat{V}^b_{ab} = g^{-1}_{ab}\,\dot{g}_{ab} = \begin{bmatrix} R^T_{ab}\dot{R}_{ab} & R^T_{ab}\dot{d}_{ab} \\ 0^T & 0 \end{bmatrix}$$

$$\boxed{\hat{V}^b_{ab} = \begin{bmatrix} \hat{\omega}^b_{ab} & \bar{v}^b_{ab} \\ 0^T & 0 \end{bmatrix}}$$

$$\text{OR} \quad \bar{V}^b_{ab} = \left(\hat{V}^b_{ab}\right)^\vee = \begin{bmatrix} \bar{v}^b_{ab} \\ \bar{\omega}^b_{ab} \end{bmatrix} = \begin{bmatrix} R^T_{ab}\dot{d}_{ab} \\ \bar{\omega}^b_{ab} \end{bmatrix}$$

Body velocity: velocity of body as seen in body's own frame.