Orthogonal Surface Coordinates

Proposition: Given a coordinatizable surface, S, a point $p \in S$, and any two independent vectors, $\vec{v}_1$ and $\vec{v}_2$ which are both in T_pS, there exists a coordinate system (f, U) for S such that $f_u = \vec{v}_1$ and $f_v = \vec{v}_2$. In particular, if $\vec{v}_1$ and $\vec{v}_2$ are orthogonal, then (f, U) is an orthogonal coordinate system.

Proof: Let (g, V) be any chosen coordinate system (not necessarily orthogonal), such that g_u and g_v are linearly independent. Since g_u and g_v are linearly independent, they span T_pS. Thus, there exists constants a, b, c, d such that:

$$\vec{v}_1 = ag_u + bg_v; \qquad \vec{v}_2 = cg_u + dg_v.$$

Define a new coordinate chart, U, such that:

$$U = \{(u, v) | (au + cv, bu + dv) \in V\}.$$

Define a new coordinate function, f, such that:

$$f : U \to \mathbb{R}^3 \qquad (u, v) \to g(au + cv, bu + dv)$$

By simple application of the chain rule, it can be seen that $f_u = \vec{v}_1$ and $f_v = \vec{v}_2$.

This proposition can be put to use as follows. Choose some way to coordinatize the ellipse, such as:

$$g(u, v) = \begin{bmatrix} A\cos(u)\cos(v) \\ B\sin(u)\cos(v) \\ C\sin(v) \end{bmatrix}$$

Let $\vec{v}_1 = g_u$ and $N = g_u \times g_v$. We can choose a vector $\vec{v}_2$ as follows:

$$\vec{v}_2 = N \times g_u = (g_u \times g_v) \times g_u = g_u \times (g_v \times g_u) = (g_u \cdot g_u)g_v - (g_u \cdot g_v)g_u.$$

Thus, in the proof of the above proposition, we choose the constants $a = 1$, $b = 0$, $c = -(g_u \cdot g_v)$, and $d = (g_u \cdot g_u)$.

NOTE: If you wish (to make your lives simpler), you may assume that $A = B$ in the above ellipse parametrization. In this case, an orthogonal coordinate system is easy to find.

Alternatively, one might be lucky to realized that an ellipsoid with principle dimensions 2A, 2B, 2C (with $A > B > C$) can be covered by the 8 *orthogonal* coordinate patches:

$$f(u, v) = \begin{bmatrix} \pm A\sqrt{\frac{(A-u)(A-v)}{(A-B)(A-C)}} \\ \pm B\sqrt{\frac{(B-u)(B-v)}{(B-A)(B-C)}} \\ \pm C\sqrt{\frac{(C-u)(C-v)}{(C-A)(C-B)}} \end{bmatrix}$$

where $u \in [C, B]$, $v \in [B, A]$. It can be shown that these coordinates are indeed orthogonal.