The Planar Contact Equations

These notes derive the basic contact equations for planar bodies in "roll-slide" contact. These equations are useful when analyzing the grasps of contacting planar bodies. They can also be used to understand several issues in the operation of gears and the design of cams.

1 The Basic Setup

Figure 1 shows the basic set up for this problem. Assume that two bodies, labeled body #1 and body #2, are in contact at a point P. Assume that body fixed reference frames are defined for each body. Think of the boundary of each object as defining a closed planar curve. Let s_1 and s_2 be the curve parameters for the boundaries of objects #1 and #1. s_1 and s_2 need not be arc-length parameters. Let $\vec{p}_1(s_1)$ and $\vec{p}_1(s_2)$ denote the functions that describes the boundaries of the bodies in their respective object frames. We call the pair (s_1, s_2) the *contact parameters* of this system.

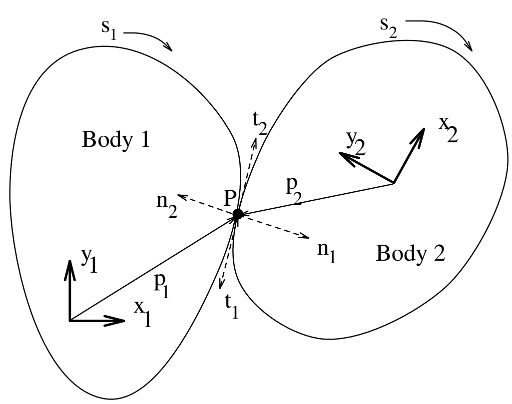

Figure 1: Geometry of contact planar bodies

Let d_{12} denote the position of the origin of body fixed frame #2 with respect to the origin of frame #1. Similarly, let R_{12} denote the rotation matrix that described the orientation of the second frame with respect to the first.

Recall that $\partial\vec{p}_i(s_i)/\partial s_i$ is the tangent vector to the i^{th} object at point s_i. Let M_i denote the length of the tangent vector:

$$M_i(s_i) = ||\frac{\partial\vec{p}_i(s_i)}{\partial s_i}|| \qquad (1)$$

2 The Constraint Equations

We wish to consider relative motions of the two bodies such that they maintain contact. There are three constraints that define an acceptable contact between the two bodies.

The point of contact constraint: The contact point on each object must be at the same location with respect to a given coordinate frame. To an observer in the first coordinate frame, the contact points can be equated as:

$$\vec{p}_1(s_1) = \vec{d}_{12} + R_{12}\vec{p}_2(s_2) \tag{2}$$

The Normal Contact Constraint: The point contact constraint expressed above is not sufficient to guarantee physically meaningful contact between the two bodies. In addition, the surface normals, $\vec{n}_1(s_1)$ and $\vec{n}_2(s_2)$, of the two bodies must point in opposite directions:

$$\vec{n}_1(s_1) = -R_{12}\vec{n}_2(s_2) \tag{3}$$

The Tangent Constraint: Similary, the tangent vectors of each body must also point in opposite directions:

$$\vec{t}_1(s_1) = -R_{12}\vec{t}_2(s_2) \tag{4}$$

where $\vec{t}_1(s_1)$ and $\vec{t}_2(s_2)$ are the unit length tangent vectors to bodies 1,2. In terms of the variables that we have introduced above, this equation can be expressed as:

$$M_1^{-1}(s_1)\frac{d\vec{p}_1}{ds_1}(s_1) = -M_2^{-1}(s_2)R_{12}\frac{d\vec{p}_2}{ds_2}(s_2) \tag{5}$$

3 An Expression for Curvature

Before proceeding, let's determine an expression, in terms of the variable introduced above, for the curvatures of the objects at the contact point. Let ρ_i be the "true" (but possibly unknown) arc-length parameter for the curve that defines the boundary of the i^{th} object. Recall that in the arc-length parameterization,

$$\frac{d^2\vec{p}_i(\rho_i)}{d\rho_i^2} = \kappa_i(\rho_i)\vec{n}_i(\rho_i) = \frac{d\vec{t}_i(\rho_i)}{d\rho_i}$$

where κ_i is the curvature of the i^{th} body and t_i is the unit tangent vector to the i^{th} body at the point of contact. Since $\vec{n}_i$ is a unit length vector,

$$\kappa_i(\rho_i) = \kappa_i\vec{n}_i \cdot \vec{n}_i = \frac{d\vec{t}_i}{d\rho_i} \cdot \vec{n}_i \tag{6}$$

Recall that the normal and tangent vectors are orthogonal: $\vec{t}_i \cdot \vec{n}_i = 0$. Taking the derivative of this equation yields:

$$\frac{d\vec{t}_i}{d\rho_i} \cdot \vec{n}_i + \vec{t}_i \cdot \frac{d\vec{n}_i}{d\rho_i} = 0 \tag{7}$$

Making use of Equations 6 7, we find that:

$$\begin{aligned}\kappa_i &= \tfrac{d\vec{t}_i}{d\rho_i}\cdot\vec{n}_i = -\vec{t}_i\cdot\tfrac{d\vec{n}_i}{d\rho_i} \\ &= -M_i^{-1}\tfrac{d\vec{p}_i}{\partial s_i}\cdot\left(\tfrac{d\vec{n}_i}{ds_i}\tfrac{ds_i}{d\rho_i}\right) \\ &= -M_i^{-2}\left(\tfrac{d\vec{p}_i}{\partial s_i}\cdot\tfrac{d\vec{n}_i}{ds_i}\right)\end{aligned} \tag{8}$$

where it should be noted that $ds_i/d\rho_i = M_i^{-1}$.

4 Detailed Derivation of the Contact Equations

The three contact constraint equations define complex nonlinear relationships between the contact parameters (s_1, s_2) and the parameters describing the relative location of the two bodies. Our goals is to related the change in the contact parameters, $(\dot{s}_1, \dot{s}_2)$, to the relative velocity of the two bodies. This relationship can be derived by taking the derivatives of the contact constraints. The details are carried out below.

Differentiate the point contact and contact normal constraints (Equations 2 and 3) with respect to time:

$$\frac{d\vec{p}_1}{ds_1}\dot{s}_1 = \dot{\vec{d}}_{12} + \dot{R}_{12}\vec{p}_2 + R_{12}\frac{d\vec{p}_2}{ds_2}\dot{s}_2 \tag{9}$$

$$\frac{d\vec{n}_1}{ds_1}\dot{s}_1 = -\dot{R}_{12}\vec{n}_2 - R_{12}\frac{d\vec{n}_2}{ds_2}\dot{s}_2 \tag{10}$$

Take the dot product of Equation 9 with $d\vec{p}_1/ds_1$ and solve for $\dot{s}_1$:

$$\dot{s}_1 = M_1^{-2}\left[\frac{d\vec{p}_1}{ds_1}\cdot\left(\dot{\vec{d}}_{12} + \dot{R}_{12}\vec{p}_2 + R_{12}\frac{d\vec{p}_2}{ds_2}\dot{s}_2\right)\right] \tag{11}$$

Substitute Equation 11 for $\dot{s}_1$ into Equation 10:

$$\frac{d\vec{n}_1}{ds_1}\left[\frac{d\vec{p}_1}{ds_1}\cdot\left(\dot{\vec{d}}_{12} + \dot{R}_{12}\vec{p}_2 + R_{12}\frac{d\vec{p}_2}{ds_2}\dot{s}_2\right)\right]M_1^{-2} = -\dot{R}_{12}\vec{n}_2 - R_{12}\frac{d\vec{n}_2}{ds_2}\dot{s}_2 \tag{12}$$

Rearrange this equation to group all of the $\dot{s}_2$ terms on the left side of the equality, and then take the dot product of both sides of the equation with $\vec{t}_1 = M_1^{-1}\frac{d\vec{p}_1}{ds_1}$:

$$\begin{aligned}&\left[M_1^{-1}\tfrac{d\vec{p}_1}{ds_1}\cdot(R_{12}\tfrac{d\vec{n}_2}{ds_2}) + M_1^{-3}\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{n}_1}{ds_1}\right)\tfrac{d\vec{p}_1}{ds_1}\cdot(R_{12}\tfrac{d\vec{p}_2}{ds_2})\right]\dot{s}_2 \\ &\quad = -M_1^{-1}\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{R}_{12}\vec{n}_2) - M_1^{-3}\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{n}_1}{ds_1}\right)\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12} + \dot{R}_{12}\vec{p}_2)\end{aligned} \tag{13}$$

Let's focus on the left hand side of this equation. Using Equation 5, the first term of the left hand side can be rewritten as:

$$M_1^{-1}\frac{d\vec{p}_1}{ds_1}\cdot(R_{12}\frac{d\vec{n}_2}{ds_2}) = -M_2^{-1}\frac{d\vec{p}_2}{ds_2}\cdot\frac{d\vec{n}_2}{ds_2} \tag{14}$$

The second term of the left hand side of Equation 13 can also be rewritten with the aid of Equation 5:

$$\begin{aligned} M_1^{-3}\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{n}_1}{ds_1}\right)\tfrac{d\vec{p}_1}{ds_1}\cdot(R_{12}\tfrac{d\vec{p}_2}{ds_2}) &= -M_1^{-3}\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{n}_1}{ds_1}\right)M_1M_2^{-1}\tfrac{d\vec{p}_2}{ds_2}\cdot\tfrac{d\vec{p}_2}{ds_2}) \\ &= -M_1^{-2}M_2\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{n}_1}{ds_1}\right) \end{aligned} \tag{15}$$

Hence, using the definition of curvature in Equation 8 and the results of Equations 14 and 15, the left hand side of Equation 13 can be expressed as:

$$-\left[M_2^{-2}\frac{d\vec{p}_2}{ds_2}\cdot\frac{d\vec{n}_2}{ds_2}+M_1^{-2}\left(\frac{d\vec{p}_1}{ds_1}\cdot\frac{d\vec{n}_1}{ds_1}\right)\right]M_2\dot{s}_2=(\kappa_1+\kappa_2)M_2\dot{s}_2.$$

The right hand side of Equation 13 can also be simplified using the definition of curvature:

$$\begin{aligned} &-M_1^{-1}\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{R}_{12}\vec{n}_2)-M_1^{-3}\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{n}_1}{ds_1}\right)\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12}+\dot{R}_{12}\vec{p}_2) \\ = \quad &+M_1^{-1}\left[-\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{R}_{12}\vec{n}_2)+\kappa_1\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12}+\dot{R}_{12}\vec{p}_2)\right] \end{aligned}$$

With these simplifications, Equation 13 can be solved for $\dot{s}_2$:

$$\dot{s}_2=(\kappa_1+\kappa_2)^{-1}M_2^{-1}M_1^{-1}\left[-\frac{d\vec{p}_1}{ds_1}\cdot\dot{R}_{12}\vec{n}_2+\kappa_1\frac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12}+\dot{R}_{12}\vec{p}_2)\right] \tag{16}$$

Let's define the variables:

$$\dot{\theta}_{12}=M_1^{-1}\frac{d\vec{p}_1}{ds_1}\cdot(\dot{R}_{12}\vec{n}_2) \tag{17}$$

$$v_t=M_1^{-1}\frac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12}+\dot{R}_{12}\vec{p}_2) \tag{18}$$

Then, Equation 16 is simply:

$$\dot{s}_2=(\kappa_1+\kappa_2)^{-1}M_2^{-1}\left[-\dot{\theta}_{12}+\kappa_1v_t\right] \tag{19}$$

Using Equations 5 and 19, Equation 11 can be expressed as:

$$\begin{aligned} \dot{s}_1 &= M_1^{-2}\left[\tfrac{d\vec{p}_1}{ds_1}\cdot\left(\dot{\vec{d}}_{12}+\dot{R}_{12}\vec{p}_2+R_{12}\tfrac{d\vec{p}_2}{ds_2}\dot{s}_2\right)\right] \\ &= M_1^{-2}\left[\tfrac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12}+\dot{R}_{12}\vec{p}_2)-M_2M_1^{-1}\left(\tfrac{d\vec{p}_1}{ds_1}\cdot\tfrac{d\vec{p}_1}{ds_1}\right)(\kappa_1+\kappa_2)^{-1}M_2^{-1}(-\dot{\theta}_{12}+\kappa_1v_t)\right] \\ &= M_1^{-1}v_t-M_1^{-1}(\kappa_1+\kappa_2)^{-1}(-\dot{\theta}_{12}+\kappa_1v_t) \\ &= M_1^{-1}(\kappa_1+\kappa_2)^{-1}(\dot{\theta}_{12}+\kappa_2v_t) \end{aligned} \tag{20}$$

Equations 20 and 19 are known as the *contact equations*:

$$\begin{aligned} \dot{s}_1 &= M_1^{-1}(\kappa_1+\kappa_2)^{-1}(\dot{\theta}_{12}+\kappa_2v_t) \\ \dot{s}_2 &= M_2^{-1}(\kappa_1+\kappa_2)^{-1}(-\dot{\theta}_{12}+\kappa_1v_t) \end{aligned} \tag{21}$$

The quantity $(\kappa_1+\kappa_2)$ is known as the *relative curvature.*

5 Interpretation of $\dot\theta_{12}$ and v_t

The physical interpretation of $\dot\theta_{12}$ should be obvious: it is the angular velocity of body #2 with respect to body #1. However, to verify this, let's evaluate the definition of $\dot\theta_{12}$ in Equation 17:

$$M_1^{-1}\frac{d\vec{p}_1}{ds_1}\cdot(\dot R_{12}\vec{n}_2) = \vec{t}_1\cdot(\dot R_{12}R_{12}^T R_{12}\vec{n}_2) = -\vec{t}_1\hat\omega_{12}^s\vec{n}_1 = -\dot\theta_{12}\vec{t}_1\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\vec{n}_1 = \dot\theta_{12} \tag{22}$$

where we have made use of Equation 3 and the fact that the tangent and normal vectors are orthogonal.

To understand v_t, let's consider the velocity of the point P in body #2 that is in contact with body #1:

$$\vec{v}_P = \dot{\vec{d}}_{12} + \dot R_{12}\vec{p}_2 = \dot{\vec{d}}_{12} + \hat\omega_{12}^s R_{12}\vec{p}_2$$

Note that

$$v_t = M_1^{-1}\frac{d\vec{p}_1}{ds_1}\cdot(\dot{\vec{d}}_{12} + \dot R_{12}\vec{p}_2) = \vec{t}_1\cdot\vec{v}_P. \tag{23}$$

Hence, v_t is the projection, onto the tangent at the contact point, of the velocity of the point in contact. It has the interpretation as the "sliding" velocity. If the second body "rolls" on the first body with no sliding, then $v_t = 0$.