1 The Moore-Penrose Pseudo Inverse

The Moore-Penrose pseudo-inverse is a general way to find the solution to the following system of linear equations:

$$\vec{b} \;=\; A\,\vec{y} \qquad\qquad \vec{b} \in \mathbb{R}^m; \quad \vec{y} \in \mathbb{R}^n; \quad A \in \mathbb{R}^{m \times n}. \tag{1}$$

Moore and Penrose showed that there is a general solution to these equations (which we will term the Moore-Penrose solution) of the form $\vec{y} = A^\dagger \vec{b}$. The matrix $A^\dagger$ is the Moore-Penrose "pseudo-inverse," and they proved that this matrix is the unique matrix that satisfies the following properties:

1. $A\,A^\dagger\,A \;=\; A$
2. $A^\dagger\,A\,A^\dagger \;=\; A^\dagger$
3. $(A\,A^\dagger)^T \;=\; A\,A^\dagger$
4. $(A^\dagger\,A)^T \;=\; A^\dagger\,A$

The Moore-Penrose pseudo-inverse and solution has the following properties. When:

- $m = n$, $A^\dagger = A^{-1}$ if A is full rank. The pseudo-inverse for the case where A is not full rank will be considered below.
- $m > n$ (which corresponds to a kinematically insufficient manipulator), the solution is the one that minimizes the quantity

  $$||\vec{b} - A\vec{y}||.$$

  That is, in this case there are more constraining equations than there are free variables $\vec{y}$. Hence, it is not generally possible find a solution to these equations. The pseudo-inverse gives the solution $\vec{y}$ such that $A^\dagger \vec{y}$ is "closest" (in a least-squared sense) to the desired solution vector $\vec{b}$.
- $m < n$ (which corresponds to a kinematically redundant manipulator), then the Moore-Penrose solution minimizes the 2-norm of $\vec{y}$: $||\vec{y}||$. In this case, there are generally an infinite number of solutions, and the Moore-Penrose solution is the particular solution whose vector 2-norm is minimal.

For application to redundant robot manipulators, we are concerned with the case where $m < n$. To understand the Moore-Penrose solution in more detail, first recall that the *Null Space* of a matrix A, denoted $\mathcal{N}(A)$, is defined as follows:

$$\mathcal{N}(A) \;=\; \{\vec{v} \mid A\vec{v} = 0\}.$$

If r is the rank of matrix A, then the null space is a linear vector space with dimension

$$\dim(\mathcal{N}(A)) \;=\; max\{0, (r-n)\}.$$

The *Row Space* of A, denoted $Row(A)$, is the linear span of its rows. Clearly, every element in $\mathcal{N}(A)$ is orthogonal to any element in $Row(A)$, and hence we say that $\mathcal{N}(A)$ and $Row(A)$ are othogonal to each other. Thus, any vector $\vec{y} \in \mathbb{R}^n$ can be uniquely split into its row and null space components: $\vec{y} = \vec{y}_{row} + \vec{y}_{null}$. Note that:

$$\vec{b} = A\vec{y} = A(\vec{y}_{row} + \vec{y}_{null}) = A\vec{y}_{row}.$$

Thus, any solution $\vec{y}$ to Equation (1) must contain $\vec{y}_{row}$ as a component. The difference between any two possible solutions lies in the amount of null space contribution to each solution. Since $Row(A)$ and $\mathcal{N}(A)$ are orthogonal to each other, it must be true by Pythogoras' theorem that:

$$||\vec{y}||^2 = ||\vec{y}_{row} + \vec{y}_{null}||^2 = ||\vec{y}_{row}||^2 + ||\vec{y}_{null}||^2.$$

From the claim above that the Moore-Penrose solution is the minimum norm solution, it must be true that the Moore-Penrose solution is the particular solution that has no null space component.

When A is full rank, the Moore-Penrose pseudo-inverse can be directly calculated as follows:

- **case** $m < n$: $A^\dagger = A^T(A\ A^T)^{-1}$
- **case** $m > n$: $A^\dagger = (A^T\ A)^{-1}A^T$

However, when A is not full rank, then these formulas can not be used. More generally, the pseudo-inverse is best computed using the *Singular Value Decomposition* reviewed below.

2 The Singular Value Decomposition

Let $A \in \mathbb{R}^{m\times n}$. Then there exists orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that the matrix A can be decomposed as follows:

$$A = U\ \Sigma\ V^T \tag{2}$$

where Σ is an $m \times n$ diagonal matrix having the form:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_p & 0 \end{bmatrix} \tag{3}$$

and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0. \qquad p = \min\{m, n\}$$

The $\{\sigma_i\}$ are termed the *singular values* of the matrix A. The columns of U are termed the *left singular vectors*, while the columns of V are termed the *right singular vectors*. The decomposition described in (2) is called the "Singular Value Decomposition," which is conveniently abbreviated as SVD.

Geometrically, the singular values of A are the lengths of the semi-axes of the hyperellipsoid E defined by

$$E = \{\vec{z} \mid \vec{z} = A\vec{x}; \quad ||\vec{x}|| = 1\}.$$

Using the SVD, the pseudo-inverse of a matrix can be easily computed as follows. Let A be decomposed as in Equation (2). Then

$$A^\dagger = V\Sigma^\dagger U^T$$

where the matrix $\Sigma^\dagger$ takes the form:

$$\Sigma = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{1}{\sigma_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{\sigma_p} & 0 \end{bmatrix} \tag{4}$$

for all of the non-zero singular values. If any of the σ_i are zero, then a zero is placed in corresponding entry of $\Sigma^\dagger$. If the matrix A is rank deficient, then one or more of its singular values will be zero. Hence, the SVD provides a means to compute the pseudo-inverse of a singular matrix.

The computation of the SVD is a non-trivial issue. It suffices to know that all respectable software packages for doing mathematics (such as maple, matlab, or mathematica) contain functions for computing the SVD. For our purposes, the existence of these procedures and the minimal facts outlined above should suffice.