Computing the Screw Parameters of a Rigid Body Displacement

Problem Statement:

We wish to determine the screw displacement parameters, for a spatial displacement. These parameters consist of:

$$\begin{aligned}
\phi &= \text{the angle of rotation about the screw axis} \\
d^{||} &= \text{the translation along the screw axis} \\
\vec{\omega} &= \text{A unit vector parallel to the screw axis} \\
\vec{\rho} &= \text{a vector to a point on the screw axis}
\end{aligned}$$

Assume that we have a rigid body which contains three non-colinear points: P, Q, R. Let P_0, Q_0, and R_0 denote the positions of the points in the body before displacement. Let P_1, Q_1, and R_1 the position of these points after a screw displacement.

The Solution:

To determine the screw parameters from the displacement of these three points, we will solve the following three simulataneous equations:

$$P_1 - P_0 = \tan(\frac{\phi}{2})\ \vec{\omega} \times (P_1 + P_0 - 2\vec{\rho}) + d^{||}\vec{\omega} \tag{1}$$

$$Q_1 - Q_0 = \tan(\frac{\phi}{2})\ \vec{\omega} \times (Q_1 + Q_0 - 2\vec{\rho}) + d^{||}\vec{\omega} \tag{2}$$

$$R_1 - R_0 = \tan(\frac{\phi}{2})\ \vec{\omega} \times (R_1 + R_0 - 2\vec{\rho}) + d^{||}\vec{\omega} \tag{3}$$

where each equation is the **Rodriguez displacement equation** for the respective points P, Q, and R.

Step #1: Subtract Equation (3) from Equations (1) and (2):

$$(P_1 - P_0) - (R_1 - R_0) = \tan(\frac{\phi}{2})\ \vec{\omega} \times [(P_1 + P_0) - (R_1 + R_0)] \tag{4}$$

$$(Q_1 - Q_0) - (R_1 - R_0) = \tan(\frac{\phi}{2})\ \vec{\omega} \times [(Q_1 + Q_0) - (R_1 + R_0)] \tag{5}$$

Form the cross product of $[(Q_1 - Q_0) - (R_1 - R_0)]$ with Equation (5):

$$\begin{aligned}
[(Q_1 - Q_0) \quad & -(R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)] \\
&= \tan(\tfrac{\phi}{2})[(Q_1 - Q_0) - (R_1 - R_0)] \times \{\vec{\omega} \times [(P_1 + P_0) - (R_1 + R_0)]\}
\end{aligned} \tag{6}$$

Note: from Equation (5), we know that $[(Q_1 - Q_0) - (R_1 - R_0)]$ is perpendicular to $\vec{\omega}$, since it results from the cross produce of a vector with $\vec{\omega}$. Therefore, the right hand side of Equation (6) will be a vector proportional to $\vec{\omega}$

We can use the vector identity $\vec{a} \times (\vec{b} \times c) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$ to simplify Equation (6):

$$\begin{aligned}[(Q_1 - Q_0) - (R_1 - R_0)] \quad & \times[(P_1 - P_0) - (R_1 - R_0)] \\ & = \tan(\tfrac{\phi}{2})[(Q_1 - Q_0) - (R_1 - R_0)] \cdot [(P_1 + P_0) - (R_1 + R_0)]\vec{\omega}\end{aligned} \tag{7}$$

We can solve Equation (7) for $\tan(\frac{\phi}{2})\vec{\omega}$:

$$\tan(\frac{\phi}{2})\vec{\omega} = \frac{[(Q_1 - Q_0) - (R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)]}{[(Q_1 - Q_0) - (R_1 - R_0)] \cdot [(P_1 + P_0) - (R_1 + R_0)]} \tag{8}$$

Thus, the rotation angle, $\tan(\frac{\phi}{2})$ can be computed as the norm to the vector in Equation (8), while $\vec{\omega}$ is the normalized vector of Equation (8).

Step #2: Now take the cross product of $\vec{\omega}$ with equation (1) and use the aforementioned vector cross product identity:

$$\begin{aligned}\vec{\omega} \times (P_1 - P_0) \quad & = \vec{\omega} \times [\tan(\tfrac{\phi}{2})\ \vec{\omega} \times (P_1 + P_0 - 2r\vec{ho}) + d^{||}\vec{\omega}] \\ & = \tan(\tfrac{\phi}{2})\ [[\vec{\omega} \cdot (P_1 + P_0)]\vec{\omega} - (P_0 + P_1) - 2(\vec{\omega} \cdot \vec{\rho})\vec{\omega} + 2\rho]\end{aligned} \tag{9}$$

Note that $\rho - (\vec{\omega} \cdot \vec{\rho})\vec{\omega} = \vec{\rho}_\perp$, where $\vec{\rho}_\perp$ is the component of $\vec{\rho}$ which is perpendicular to $\vec{\omega}$. That is, while $\vec{\rho}$ is a vector from the origin of the reference frame to *any* point on the screw axis, $\vec{\rho}_\perp$ is the shortest vector to the point on the screw axis closest to the origin of the reference frame. Equation (9) can then be solved for $\vec{\rho}_\perp$:

$$\vec{\rho}_\perp = \frac{1}{2}\left[\frac{\vec{\omega} \times (P_1 - P_0)}{\tan\frac{\phi}{2}} - (\vec{\omega} \cdot (P_1 + P_0))\vec{\omega} + P_0 + P_1\right] \tag{10}$$

Step 3: Finally, we can use Equation (1), (2), or (3) to find $d^{||}$:

$$d^{||} = \vec{\omega} \cdot (P_1 - P_0) = \vec{\omega} \cdot (Q_1 - Q_0) = \vec{\omega} \cdot (R_1 - R_0) \tag{11}$$