Example: planar contact equations

Here we will derive the contact equations for an ellipse rolling/sliding on a flat surface. Recall that the boundary of the ellipse can be parametrized as:

$$\begin{bmatrix} x(\mu_1) \\ y(u) \end{bmatrix} = \begin{bmatrix} a\cos(\mu_1) \\ b\sin(\mu_1) \end{bmatrix}$$

where μ_1 is the "curve parameter" of the ellipse boundry. It is not necessarily the arc-length parameter. Note, with this parametrization, the normal vector (see below) is pointing *inward* into the object. Note that when $a = b$, the ellipse is a circle.

Let's define the object frames on the two bodies as follows:

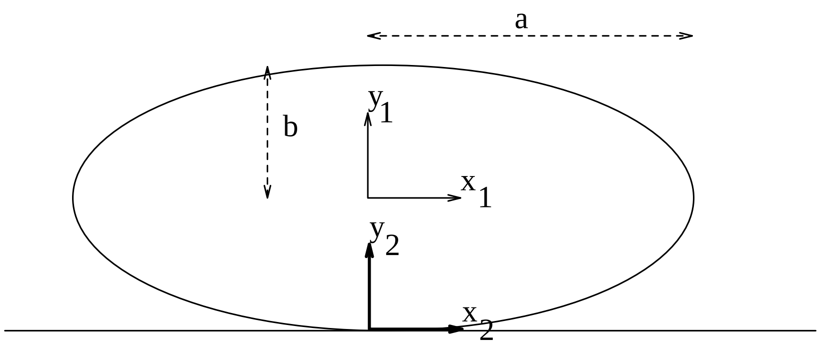

Figure 1: Planar Ellipse

With the coordinate frame chosen in Fig. 1, the flat surface can be parametrized as:

$$c_2(\mu_2) = \begin{bmatrix} -\mu_2 \\ 0 \end{bmatrix}$$

The negative sign in front of μ_2 means that μ_2 increases to the left. This choice arises because we must choose the surface normal vector of body 2 (the plane) to point into body 2, in order to be consistent with our choice for body 1.

Thus, the tangent vectors (not necessarily unit length) to the two surfaces are:

$$\vec{t}_1(\mu_1) = \begin{bmatrix} -a\sin\mu_1 \\ b\cos\mu_1 \end{bmatrix}; \qquad \vec{t}_2(\mu_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Similarly:

$$M_1 = |\vec{t}_1| = (a^2\sin^2\mu_1 + b^2\cos^2\mu_1)^{1/2} \tag{1}$$

$$M_2 = |\vec{t}_2| = 1 \tag{2}$$

Recall that the curvature of the i^{th} planar surface could be computed as:

$$\kappa_i = -M_i^{-2}\left(\frac{\partial c_i}{\mu_i}\right)^T \frac{\partial \vec{n}_i}{\partial \mu_i}$$

where $\vec{n}_i$ is the unit vector normal to the curve. Clearly, for the flat surface, $\kappa_2 = 0$. To compute κ_i, let us derive an expression for $\vec{n}_1$ and its derivatives. For planar curves, the normal can be easily computed as the unit tangent vector rotated by $\pm\pi/2$

$$\vec{n}_1 = -\frac{1}{(a^2\sin^2\mu_1 + b^2\cos^2\mu_1)^{1/2}}\begin{bmatrix} b\cos\mu_1 \\ a\sin\mu_1 \end{bmatrix}$$

Consequently:

$$\frac{\partial \vec{n}_i}{\partial \mu_1} = -\frac{ab}{(a^2\sin^2\mu_1 + b^2\cos^2\mu_1)^{3/2}}\begin{bmatrix} -a\sin\mu_1 \\ b\cos\mu_1 \end{bmatrix}$$

Thus:

$$\kappa_i = \frac{ab}{(a^2\sin^2\mu_1 + b^2\cos^2\mu_1)^{3/2}}$$

Assuming that v_t and $\dot{\theta}$ are given, the contact equations are:

$$\dot{\mu}_1 = (\kappa_1+\kappa_2)^{-1}M_1^{-1}(-\dot{\theta}+\kappa_2 v_t) = -\frac{\dot{\theta}}{\kappa_1 M_1} = -\frac{(a^2\sin^2\mu_1 + b^2\cos^2\mu_1)}{ab}\dot{\theta} \quad (3)$$

$$\dot{\mu}_2 = (\kappa_1+\kappa_2)^{-1}M_2^{-1}(\dot{\theta}+\kappa_1 v_t) = v_t + \frac{\dot{\theta}}{\kappa_1} \quad (4)$$

Note: if $\dot{\theta} = 0$, then $\dot{\mu}_1 = 0$. That is, the point of contact on the ellipse does not move if it purely slides along the flat surface. This meets with intuition. Also note that if $v_t = 0$, and $\dot{\theta} > 0$, then $\dot{\mu}_1 < 0$. That is, counter-clockwise rotation of the object causes a clockwise rotation of the contact point on the surface of the ellipse. This too checks with our intuition.

Now consider the $\dot{\mu}_2$ equation. If $\dot{\theta} > 0$, i.e. counter-clockwise rotation of body 1, then $\dot{\mu}_2 > 0$. This means that the contact point moves to the left. This also checks with our intuition. What if $v_t > 0$ and $\dot{\theta} = 0$?. In this case, $\dot{\mu}_2 > 0$, implying that the contact point moves to the left. However, recall that v_t is defined as the projection of the velocity of the contact point onto the tangent vector of body 2. With the parametrization of body 2 introduced above, the tangent vector of the plane is pointing to the left. Thus, $v_t > 0$ implies that the body moves to the left. Thus, this result also checks with our intuition.

The contacts equations for the case that the ellipse roll/slides on a circle, instead of a flat plane, are similar, except the curvature of the circle is $1/R$, while the plane had zero curvature. The relative curvature will only zero if the ellipse is rolling on the *inside* of a circle, and the ellipse curvature equals $1/R$. This is highly unlikely, since, as this would often imply that ellipse would intersect the circle elsewhere (i.e., the two bodies would penetrate each other).