1 The Differential Geometry of Surfaces

Three-dimensional objects are bounded by surfaces. This section reviews some of the basic definitions and concepts relating to the geometry of smooth surfaces.

1.1 Manifolds

Before considering the surfaces that bound 3-dimensional solid objects, we first pause to consider the concept of a *manifold*.

Definition 1 *Let X and Y be subsets of two Euclidean spaces and let $f\colon X \to Y$ be bijective (i.e., surjective and one-to-one). If f and f^{-1} are continuous, then f is a* homeomorphism*. If f and f^{-1} are smooth, then f is a* diffeomorphism*.*

Definition 2 *A k-dimensional* manifold*, M, is locally diffeomorphic to $\mathbb{R}^k$. That is, for each point $x \in M$, there exists a neighborhood $\mathcal{V} \subset M$ which is diffeomorphic to an open set $\mathcal{U} \subset \mathbb{R}^k$.*

1.2 Surfaces

The concept of a manifold provides us with a general notion of a surface. For dealing with surfaces that bound 3-dimensional bodies, we will want to add some additional structure.

Definition 3 *A* coordinatizable surface*, $\mathcal{S}$, is the image of a map $f\colon\ U \to \mathbb{R}^3$ where*

i) U is an open connected subset of $\mathbb{R}^2$.

ii) The vectors $\frac{\partial f}{\partial u}(u,v)$ and $\frac{\partial f}{\partial v}(u,v)$ are linearly independent for all $(u,v) \in U$.

iii) f is a homeomorphism.

We say that (f, U) is a *coordinate system* for S with coordinates u, v. The function f^{-1} is termed a *local parametrization*. The coordinate system is *orthogonal* if $\frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} = 0$.

Example 1 **Sphere.** *One coordinate systems for the sphere consists of:*

$$\mathcal{U} = \{(u,v) \mid -\frac{\pi}{2} < u < \frac{\pi}{2}; -\pi < v < \pi\} \qquad f(u,v) = \begin{bmatrix} \cos(u)\cos(v) \\ -\cos(u)\sin(v) \\ \sin(v) \end{bmatrix}.$$

It can be verified that $\frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} = 0$, thereby implying that our chosen coordinate system is an orthogonal one.

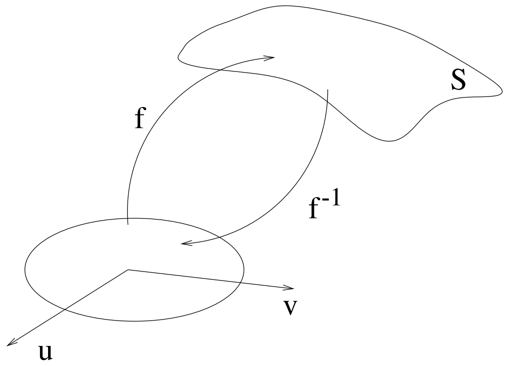

Figure 1: Coordinatizable Surface

Let S be a coordinatizable surface and let $(f, \mathcal{U})$ be a coordinate system for S in the neighorhood of a point $p \in S$. Without loss of generality, assume that $f(0,0) = p$.

Definition 4 *The*tangent space *to S at $p \in S$, denoted T_pS, is the image of $df(0,0)$:*

$$T_pS = df_{(0,0)}(\mathbb{R}^k) = \{df(0,0)\mathbf{v} \mid \forall\ \mathbf{v} \in \mathbb{R}^k\}$$

Example 2 *(***Sphere** *continued). Let $p = f(0,0) = [1\ 0\ 0]^T$. This point is located where the x-axis intersects the surface of the sphere. Then*

$$df_{(0,0)} = \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This plane is spanned by vectors parallel to the y and z-axes. Hence, T_pS can be viewed as the plane passing through p and parallel to the y-z plane.

Remarks:

1. The tangent space, T_pS, is the closest linear approximation to S at p. That is, df is the linear term in the Taylor Series expansion of f at $f(0,0)$.
2. In general, if $p_1 \neq p_2$, then $T_{p_1}S \neq T_{p_2}S$.
3. The *dimension* of a manifold, M, is defined as the dimension of its tangent space: $dim(M) = dim(T_pS)$.
4. The definition of the tangent space is intrinsic. I.e., it does not depend upon the choice of coordinate system.

A *tangent vector* at $p \in S$ is a vector in T_pS. Tangent vectors can also be viewed in the following way. Let $\alpha(t)$ be a parametrized curve lying in S such that $\alpha(0) = p$. Hence,

$$\alpha(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x((u(t), v(t)) \\ y(u(t), v(t)) \\ z(u(t), v(t)) \end{bmatrix} = f(u(t), v(t)).$$

A tangent vector at $p \in S$ can also be thought of as the tangent to the curve $\alpha(t)$ at $\alpha(0)$:

$$\frac{d\alpha}{dt}(t) = \frac{\partial \alpha}{\partial u}\frac{du}{dt} + \frac{\partial \alpha}{\partial v}\frac{dv}{dt} = f_u u^{'} + f_v v^{'}$$

where $f_u = \frac{\partial f}{\partial u}$ and $f_v = \frac{\partial f}{\partial v}$.

1.3 The First Fundamental Form and the Metric Tensor

It is desirable to be able to determine properties such as distance, angle, and area on a surface, S, without referring back to the ambient space in which S is embedded.

Since T_pS is a vector space, it is possible to define an inner product on T_pS, denoted by $< \, , \, >_p$, such that if $v_1, v_2 \in T_pS$, then $< v_1, v_2 >_p$ is equal to the inner product of v_1 and v_2 as vectors in $\mathbb{R}^3$.

Definition 5 *The quadratic form $I_p(\mathbf{v}) =< \mathbf{v}, \mathbf{v} >_p$ defined on T_pS is called the* 1^{st} Fundamental Form *of S at p.*

An expression for $I_p(\mathbf{v})$ in terms of coordinates on S can be derived as follows. Let $\alpha(t)$ be a curve lying in S and passing through $p \in S$ at $t = 0$. Then:

$$\begin{aligned} I_p(\alpha^{'}(0)) &= < \alpha^{'}(0), \alpha^{'}(0) >_p =< f_u u^{'} + f_v v^{'}, f_u u^{'} + f_v v^{'} >_p \\ &= < f_u, f_u >_p (u^{'})^2 + 2 < f_u, f_v >_p u^{'} v^{'} + < f_v, f_v >_p (v^{'})^2 \\ &= \begin{bmatrix} u^{'} \\ v^{'} \end{bmatrix}^T \begin{bmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{bmatrix} \begin{bmatrix} u^{'} \\ v^{'} \end{bmatrix} \end{aligned}$$

Example 3 **The sphere** *(continued).*

$$f_u = \begin{bmatrix} -\sin(u)\cos(v) \\ \sin(u)\sin(v) \\ \cos(u) \end{bmatrix} \qquad f_v = \begin{bmatrix} -\cos(u)\sin(v) \\ -\cos(u)\cos(v) \\ 0 \end{bmatrix}$$

Consequently, $f_u \cdot f_u = 1$, $f_u \cdot f_v = 0$, and $f_v \cdot f_v = \cos^2(u)$. Hence, $I_p = (u^{'})^2 + (v^{'})^2 \cos^2(u)$.

Definition 6 *The* Metric Tensor *is the* 2×2 *matrix that satisfies the relationship:*

$$I_p = M_p M_p$$

If $(f, \mathcal{U})$ *is an orthogonal coordinate system, then*

$$M_p = \begin{bmatrix} ||f_u|| & 0 \\ 0 & ||f_v|| \end{bmatrix}$$

1.4 The Gauss Map

Let a surface, S, be parametrized in a neighborhood of $p \in S$ by $(f, \mathcal{U})$. Let's choose a *surface normal vector* at each p according to the rule:

$$N(p) = \frac{f_u \times f_v}{|f_u \times f_v|}(p)$$

where f_u and f_v are evaluated at the point (u, v) such that $f(u, v) = p$. If $\mathcal{V} \subset S$ is open in S and if $N\colon\ \mathcal{V} \to \mathbb{R}^3$ is differentiable, then we say that N is a differentiable field of unit normal vectors on $\mathcal{V}$. Some surfaces (such as a Mobius strip) do not admit a globally defined unit normal vector field. If a surface S does admit such a vector field, then we say that S is orientable. The choice of the normal field is termed an *orientation.*

Definition 7 *Let* $S \subset \mathbb{R}^3$ *be a surface with orientation* N*. The map* $N\colon\ S \to \mathbb{R}^3$ *taking values in the unit sphere* S^2 *is called the* Gauss map *of* S*.*

In general f_u need not be orthogonal to f_v. However, as shown in the following proposition, it is always possible to find an orthgonal coordinate system in the neighborhood of a given point.

Proposition 1.1 *Given a coordinatizable surface,* S*, a point* $p \in S$*, and any two independent vectors,* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *which are both in* T_pS*, there exists a coordinate system* (f, U) *for* S *such that* $f_u = \mathbf{v}_1$ *and* $f_v = \mathbf{v}_2$*. In particular, if* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are orthogonal, then* (f, U) *is an orthogonal coordinate system.*

Proof: Let (g, V) be any chosen coordinate system (not necessarily orthogonal), such that g_u and g_v are linearly independent. Since g_u and g_v are linearly independent, they span T_pS. Thus, there exists constants a, b, c, d such that:

$$\mathbf{v}_1 = ag_u + bg_v; \qquad \mathbf{v}_2 = cg_u + dg_v.$$

Define a new coordinate chart, $\mathcal{U}$, such that:

$$\mathcal{U} = \{(u, v) | (au + cv, bu + dv) \in V\}.$$

Define a new coordinate function, f, such that:

$$f\colon \mathcal{U} \to \mathbb{R}^3 \qquad (u,v) \to g(au+cv, bu+dv)$$

By simple application of the chain rule, it can be seen that $f_u = \mathbf{v}_1$ and $f_v = \mathbf{v}_2$. $\square$

Example 4 *Consider and ellipse that is coordinatized in the following nonorthogonal way:*

$$g(u,v) = \begin{bmatrix} A\cos(u)\cos(v) \\ B\sin(u)\cos(v) \\ C\sin(v) \end{bmatrix}$$

Let $\mathbf{v}_1 = g_u$ *and* $N = g_u \times g_v$. *We can choose a vector* $\mathbf{v}_2$ *as follows:*

$$\mathbf{v}_2 = N \times g_u = (g_u \times g_v) \times g_u = g_u \times (g_v \times g_u) = (g_u \cdot g_u)g_v - (g_u \cdot g_v)g_u.$$

Thus, in the proof of the above proposition, we choose the constants $a = 1$, $b = 0$, $c = -(g_u \cdot g_v)$, *and* $d = (g_u \cdot g_u)$.

Because we can always find an orthogonal coordinate system for the neighborhood of a given point, it is always possible to assign a reference frame at each surface point in this neighborhood.

Definition 8 *Let S be a coordinatizable surface and assume that $(f, \mathcal{U})$ is an orthogonal coordinate system for S in the neighborhood of $p \in S$. The* normalized Gauss frame *at $p \in S$ is the frame with origin at p and whose orthonormal basis vectors are:*

$$\mathbf{x} = \frac{f_u}{|f_u|} \qquad \mathbf{y} = \frac{f_v}{|f_v|} \qquad \mathbf{z} = N(p).$$

The Gauss frame map *is the function $g\colon \mathcal{U} \to \mathbb{R}^3 \times SO(3)$ such that*

$$(u,v) \to (f(u,v), [\mathbf{x}, \mathbf{y}, \mathbf{z}]).$$

1.5 The Second Fundamental Form and the Curvature Tensor

We next consider the differential of the Gauss map at a point $p \in S$. The differential, dN_p is a linear map from T_pS to $T_{N(p)}S^2$. In general, T_pS and $T_{N(p)}S^2$ are not the same. However, they are both two dimensional vector spaces, and when interpreted as tangent planes, they are both oriented in a parallel way. Hence we may reasonably identify these two spaces, $T_pS \simeq T_{N(p)}S^2$, and consider dN_p as a linear map from T_pS to T_pS.

To interpret, and ultimately compute, the differential of the Gauss map, consider a parameterized curve $\alpha(t)$ lying in S such that $\alpha(0) = p \in S$. The function $N(\alpha(t))$ can be interpreted as a curve of normal vectors. The tangent vector $N^{'}(\alpha(0)) = dN_p\alpha^{'}(0)$ is a vector in $T_{N(p)} \simeq T_PS$. It measures the rate of change of the normal vector at $p = \alpha(0)$ when N is restricted to the curve $\alpha(t)$. Hence, dN_p measures how the normal vector varies at p for movement along the surface S in the direction of $\alpha^{'}(0)$.

Example 5 Cylinder. *A cylinder of radius R can be given the following coordinate system:*

$$\mathcal{U} = \{(u,v)| \; -\pi < u < \pi; \; v \in \mathbb{R}\} \qquad f(u,v) = \begin{pmatrix} R\cos(u) \\ R\sin(u) \\ v \end{pmatrix}.$$

The normal to the cylinder surface is:

$$N = \frac{1}{|f_u \times f_v|} f_u \times f_v = \begin{pmatrix} R\cos(u) \\ R\sin(u) \\ 0 \end{pmatrix}$$

Let's compute dN for two different curves. First consider the curve $\alpha(t) = [R\cos u(t), R\sin u(t), 0]^T$, where $u = \omega t$ for ω a constant. The tangent to α is given by $\alpha^{'}(t) = [-R\sin(u(t)), R\cos(u(t)), 0]^T u^{'}$. For this curve

$$\frac{dN}{dt} = dN\alpha^{'} = \begin{pmatrix} -R\sin(u(t)) \\ R\cos(u(t)) \\ 0 \end{pmatrix} u^{'} = \alpha^{'}(t)$$

Consider the curve $\alpha_2(t) = [0,0,t]^T$, where $\alpha_2'(t) = [0,0,1]^T$. In this case,

$$\frac{dN}{dt} = dN\alpha^{'}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since dN_p defines how the surface normal vector varies in the neighborhood of p, we can use to to define the following important characterization of a surface.

Definition 9 *The quadratic form $II_p(\mathbf{v})$ defined by*

$$II(\mathbf{v}) = - < dN_p\mathbf{v}, \mathbf{v} >_p; \qquad v \in T_pS$$

is called the 2^{nd} fundamental form *of S at p.*

The 2^{nd} fundamental form can be interpreted as follows. Let $\alpha(s)$ be an arc-length parametrized curve lying in S, and let $\alpha(0) = p$. Let $N(s)$ denote the restriction of the Gauss map to $\alpha(s)$. Note that the surface normal vector is always orthogonal to a surface tangent vector: $< N(s), \alpha(s) >_p = 0$. Thus, taking the derivative of both sides of this relationship implies that:

$$\begin{aligned} II(\alpha^{'}(0)) &= - < N^{'}(0), \alpha^{'}(0) >_p = - < dN_p\alpha^{'}(0), \alpha^{'}(0) >_p \\ &= < N(0), \alpha^{''}(0) >_p = < N(0), \kappa(0)\mathbf{n}(0) >_p \\ &= \kappa_n(p) \end{aligned}$$

where $\mathbf{n}(s)$ is the normal vector to the curve $\alpha(s)$ and we have used the relationship $\alpha^{''}(s) = \kappa(s)\mathbf{n}(s)$ for the curve $\alpha(s)$. We term the quantity $\kappa_n(p)$ the *normal curvature* at p. We can interpret the normal curvature as follows. At point p, let $N(p)$ be the normal to the surface at p, and let $\alpha(s)$ be a curve passing through p, whose normal vector is $\mathbf{n}(s)$. The normal curvature is equal to $\kappa\cos(\theta)$, where θ is the angle between N and $\mathbf{n}$. The normal curvature also has another interpretation. Let a *normal section* be the intersection of the surface S with a plane containing N, $\alpha^{'}(0)$, and p. In a neighborhood of p, the normal section formed by this intersection is a regular plane curve. The curvature of this curve at p is equal to the absolute value of κ_n. Note that *all* curves passing through x with the same tangent, $\alpha^{'}(0)$, have the same normal curvature.

Let's now derive a coordinate expression for the second fundamental form. Let $(f,\mathcal{U})$ be a coordinate system for S, and let $\alpha(t) = f(u(t), v(t))$ be a not necessarily arc-length parametrized curve lying in S. Then

$$\begin{aligned} II_p(\alpha^{'}) &= -< dN_p(\alpha^{'}), \alpha^{'} >_p = -< N_u u^{'} + N_v v^{'}, N_u u^{'} + N_v v^{'} >_p \\ &= -\begin{bmatrix} u^{'} & v^{'} \end{bmatrix}^T \begin{bmatrix} (f_u \cdot N_u) & (f_u \cdot N_v) \\ (f_v \cdot N_u) & (f_v \cdot N_v) \end{bmatrix}_p \begin{bmatrix} u^{'} \\ v^{'} \end{bmatrix} \\ &= -\begin{bmatrix} u^{'} & v^{'} \end{bmatrix}^T \mathcal{II}_p \begin{bmatrix} u^{'} \\ v^{'} \end{bmatrix} \end{aligned}$$

where $u^{'} = du/dt$, $f_u = \partial f/\partial t$, $N_u = \partial N/\partial u$, etc.

Thus, $II(\mathbf{v})$ indicates how the surface normal varies when moving on the surface in the tangent direction $\mathbf{v}$.

We now consider a geometrical quantity, the *Curvature Tensor*, that is closed related to the second fundamental form. Let's assume that a surface S can be locally parametrized in the vicinity of a point p by an orthogonal coordinate system.

Definition 10 *Let S be a coordinatizable surface with orthogonal coordinate system. The* Curvature Tensor*, $\mathcal{K}$, at point $p \in S$ is the matrix*

$$\mathcal{K} = M_p^{-1} \mathcal{II}_p M_p^{-1} = \begin{bmatrix} \frac{f_u \cdot N_u}{|f_u|^2} & \frac{f_u \cdot N_v}{|f_u||f_v|} \\ \frac{f_v \cdot N_u}{|f_u||f_v|} & \frac{f_v \cdot N_v}{|f_v|^2} \end{bmatrix}$$

where M_p is the metric tensor of S at p.

The use of the metric tensor in the above definition makes the curvature tensor independent of the choice of coordinate system.

The eigenvalues of $\mathcal{K}$ are called the *principal curvatures*, while the associated eigenvectors are termed the *principal axes* of curvature.

1.6 Torsion

Curvature alone is insufficient to uniquely define the local geometry of a surface. This insufficiency can be seen as follows. Assume that $(f, \mathcal{U})$ is an orthogonal coordinate system in the neighborhood of a point $p \in S$. The **Gauss frame** at p will be defined as the coordinate frame whose orthonormal basis vectors are $\frac{f_u}{|f_u|}$, $\frac{f_v}{|f_v|}$, and N. The Gauss frame is defined at each point on an orientable surface. Hence, the local properties of the surface can be deduced from the variations of the Gauss frame in the vicinity of a point. The curvature tensor determines the rotation of the Gauss from about the axes that lie in the surfaces's tangent plane. However, we need an additional concept, which we shall call *torsion*, that describes how the Gauss frame "twists" about the surface normal as the frame is moved around the surface.

Let $(f, \mathcal{U})$ be a coordinate system for a surface S, which we will assume to be orthogonal. By definition, the vector $\mathbf{x} = \frac{f_u}{|f_u|}$ will be a unit vector tangent to S. Let $\mathbf{y} = \frac{f_v}{|f_v|}$. By construction, this vector will be a unit vector that is tangent to S, orthogonal to $\mathbf{x}$, and satisfy $\mathbf{x} \times \mathbf{y} = N$, where N is the normal to S. Since $\mathbf{x}$ is a unit vector, its derivative must be orthogonal to $\mathbf{x}$. Thus, it will have a component along $\mathbf{y}$ and a component along N. The projection of this derivative on $\mathbf{y}$ will yield a measure of Gauss frame twist about N.

Let $\alpha(t)$ be a not necessarily arc-length parametrized curve lying in S. Assume that $\alpha(0) = p \in S$ and that $\alpha'(0) = \mathbf{x}$. Then:

$$
\begin{aligned}
\mathbf{y}^T \frac{d\mathbf{x}(t)}{dt} &= \left(\frac{f_v}{|f_v|}\right)^T \left[\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right] \\
&= \left(\frac{f_v}{|f_v|}\right)^T \left[\frac{\partial}{\partial u}\left(\frac{f_u}{|f_u|}\right) u' + \frac{\partial}{\partial v}\left(\frac{f_u}{|f_u|}\right) v'\right] \\
&= \left(\frac{f_v}{|f_v|}\right)^T \left[\frac{f_{uu}}{|f_u|} u' + \frac{f_{uv}}{|f_u|} v'\right] \\
&= \begin{bmatrix} \frac{f_v \cdot f_{uu}}{|f_u||f_v|} & \frac{f_v \cdot f_{uv}}{|f_u||f_v|} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}
\end{aligned}
$$

We will define the *torsion tensor* at $p \in S$ as the matrix

$$
\mathcal{T}_p = \begin{bmatrix} \frac{f_v \cdot f_{uu}}{|f_u|^2|f_v|} & \frac{f_v \cdot f_{uv}}{|f_u||f_v|^2} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix}.
$$

Hence, the torsion tensor satisfies the relationship:

$$
\mathbf{y}^T \mathbf{x}' = \mathcal{T}_p M_p \begin{bmatrix} u' \\ v' \end{bmatrix}
$$