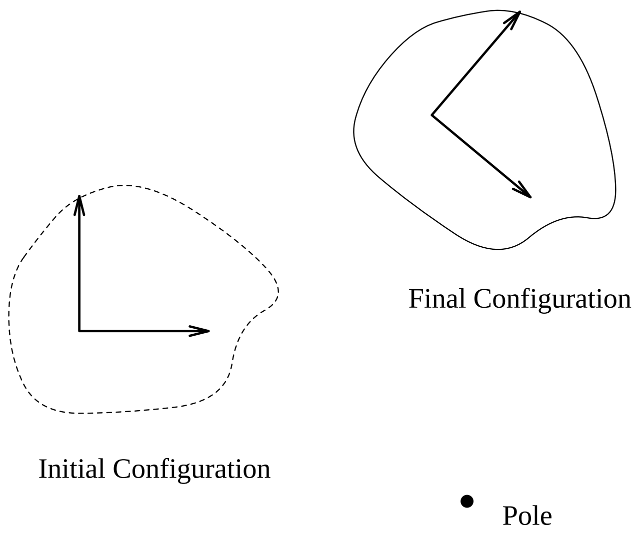
ME 115(a): Solution to Homework #1



Problem 1: There are two ways to solve this problem. In both solutions, let the displacement be defined by $D_{12} = (\vec{d}_{12}, R_{12})$

Solution 1: The first solution is based on the observation that location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position ${}^0\vec{v}$ as seen by an observer in the fixed frame. After displacement, the observer in the body fixed frame also sees the pole in his/her coordinates at point ${}^0\vec{v}$. However, the moving body has displaced relative to the fixed observer by amount $g_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by the coodinate transform:

$$^0\vec{v} = \vec{d}_{12} + R^0_{12}\vec{v}.$$

This equation can be solved to find the pole location:

$$^0\vec{v} = (I - R_{12})^{-1}\vec{d}_{12}$$

Of course, you need to show the fact that $(I - R_{12})$ is invertible. It will always be invertible, except when $R_{12} = I$. In this case, the motion is a pure translation, and the pole is the "pole at infinity."

Solution 2: The second solution proceeds as follows. Select a new, denoted 0, whose basis vectors are parallel to Frame 1 and whose origin lies at the pole of the displacement. We do not yet know the location of this pole. Let $\vec{p}$ denote the location of the pole, as seen by an observer in Frame 1. The location of Frame 1 relative to Frame 0 is a pure translation of amount $-{}^1\vec{p}$, and therefore, $D_{01} = (-\vec{p}, I)$. The displacement of the body from the first position to the second position, as now observed in Frame 0, is obtained by a similarity transform $D_{01}D_{12}D_{01}^{-1}$:

$$\begin{aligned}
D_{01}D_{12}D_{01}^{-1} &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(-\vec{p}, I)^{-1} & (1)\\
&= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(+\vec{p}, I) & (2)\\
&= (-\vec{p}, I)((\vec{d}_{12} + R_{12}\vec{p}), R_{12}) & (3)\\
&= ((\vec{d}_{12} + (R_{12} - I)\vec{p}), R_{12}) & (4)
\end{aligned}$$

Hence, if $\vec{p} = -(R_{12} - I)^{-1}\vec{d}_{12} = (I - R_{12})^{-1}\vec{d}_{12}$, then $D_{01}D_{12}D_{01}^{-1} = (\vec{0}, R_{12})$. I.e., as viewed in reference Frame 0, the displacement is a pure rotation by amount R_{12}.

Problem 2: You were asked to find the pole of the isplacement: $D_1 = (x, y, \theta) = (1.5, 2.0, 45.0^o)$. Using the above results:

$$\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(45^o) & -\sin(45^o) \\ \sin(45^o) & \cos(45^o) \end{pmatrix}\right]^{-1} \begin{bmatrix} 1.5 \\ 2.0 \end{bmatrix} = \begin{bmatrix} -1.66421 \\ 2.81066 \end{bmatrix}$$

Problem 3: You were to "prove" that a body undergoing spherical motion has three degrees of freedom.

A body undergoing spherical motion has one fixed point. Let the body consist of N particles. Let P_1 denote the particle lying at the fixed point. A point in 3-dimensional Euclidean space normally requires 3 independent variables to fix its location. However, since P_1 does not move, it actually has 0 degrees-of-freedom (DOF). Now consider a particle P_2 in the body. Particle P_2 has 3 DOF as a particle. However, it is constrained to lie a fixed distance, d_{12} from particle P_1 due to the fact that P_1 and P_2 are part of the same rigid body. The fixed distance relationship imposes one constraint on P_2. Next consider a point P_3, which lie a fixed distance from P_1 and P_2. Therefore, there are two constraints on its location. Now, consider a particle P_4. Since its must lie a fixed distance from P_1, P_2, and P_3, there are three constraints on its motion. Particles P_5, …, P_N similarly have 3 constraints.

The total number of degrees of freedom of the N particles are: $3(N-1) + 0 = 3N - 3$. The total number of constraints on these particles are: $1 + 2 + 3(N-3) = 3N - 6$. Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body: $(3N - 3) - (3N - 6) = 3$.

Problem 4: Problem 1 in chapter 2 of MLS.

You can do both of these in a brute force way. Here are some slightly less grungy alternatives.

Part (a): An easy way to solve this problem is to use the properties of the $\hat{}$ operator.

$$a \cdot (b \times c) = a^t\hat{b}c = a^t(-\hat{b}^T)c = -(\hat{b}a)^Tc = -(b \times a) \cdot c = (a \times b) \cdot c$$

Part (b): In studying the equatin $a \times (b \times c)$, assume that a, b, and c are linearly independent. If they are not, then this identity is trivial. Let $v = (b \times c)$. Since v is orthogonal to both

b and c, the triple v, b, and c span $\mathbb{R}^3$. Consequently, there exists constants α_1, α_2, and α_3 such that:

$$a = \alpha_1 v + \alpha_2 b + \alpha_3 c$$

where α_2 and α_3 can be found as:

$$\begin{aligned} a \cdot b &= \alpha_2 ||b||^2 + \alpha_3 c \cdot b && (5) \\ a \cdot c &= \alpha_2 b \cdot c + \alpha_3 ||c||^2 && (6) \end{aligned}$$

The result can then be obtained by direct calculation.

$$\begin{aligned} a \times (b \times c) &= (\alpha_1 v + \alpha_2 b + \alpha_3 c) \times (b \times c) && (7) \\ &= \alpha_2 \hat{b}\hat{b} c - \alpha_3 \hat{c}\hat{c} b && (8) \end{aligned}$$

One can derive the fact that if q is a 3×1 vector, then $\hat{q}\hat{q} = qq^T - (q \cdot q)I$, where I is the 3×3 identity matrix. Hence,

$$\begin{aligned} a \times (b \times c) &= \alpha_2 \hat{b}\hat{b} c - \alpha_3 \hat{c}\hat{c} b && (9) \\ &= \alpha_2 (bb^T - ||b||^2 I) c - \alpha_3 (cc^T - ||c||^2 I) b && (10) \\ &= (\alpha_2 b \cdot c + \alpha_3 ||c||^2) b - (\alpha_2 ||b||^2 + \alpha_3 b \cdot c) c && (11) \\ &= (a \cdot c) b - (a \cdot b) c. && (12) \end{aligned}$$