## ME 115(a): Solution to Homework #1

**Problem 1:** Recall that the location of the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position ${}^{B}\vec{p}$ as seen by an observer in the fixed $B$ frame. After displacement, the observer in the body fixed $C$ frame also sees the pole in his/her coordinates at point ${}^{B}\vec{p}$. However, the moving body has displaced relative to the fixed observer by amount $D_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by a coodinate transform. Since the pole is at the same location in both the fixed and moving frames, it must be true that:

$${}^{B}\vec{p} = \vec{d}_{12} + R_{12}\ \ {}^{B}\vec{p}.$$

This equation can be solved to find the pole location:

$${}^{B}\vec{p} = (I - R_{12})^{-1}\vec{d}_{12}$$

Of course, you need to show the fact that $(I - R_{12})$ is invertible. It will always be invertible, except when $R_{12} = I$. In this case, the motion is a pure translation, and the pole is the "pole at infinity."

To find the pole of the displacement: $D_2 = (x, y, \theta) = (1.0, 2.0, 30.0^o)$, substitute into the above results:

$${}^{B}\vec{p} = (I - R_{12})^{-1}\vec{d}_{12} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(30^o) & -\sin(30^o) \\ \sin(30^o) & \cos(30^o) \end{pmatrix}\right]^{-1} \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} = \begin{bmatrix} -3.23205 \\ 2.86603 \end{bmatrix}$$

You could report this result in Frame B, or transform the results to frame A.

$${}^{A}\vec{p} = \vec{d}_{01} + R_{01}\ \ {}^{B}\vec{p} = \begin{bmatrix} 1.0 \\ 3.0 \end{bmatrix} + \begin{pmatrix} \cos(45^o) & -\sin(45^o) \\ \sin(45^o) & \cos(45^o) \end{pmatrix} \begin{bmatrix} -3.23205 \\ 2.86603 \end{bmatrix}$$

**Problem 2:** To show that a transformation is a pure rotation when viewed in a reference frame at the pole, select a new reference frame, denoted by $D$, whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement. Let $\vec{p}$ denote the location of the pole, as seen by an observer in Frame B. The location of Frame $B$ relative to Frame $D$ is a pure translation of amount $-{}^{1}\vec{p}$, and therefore, $D_{DB} = (-\vec{p}, I)$. The displacement of the body from the first position to the second position, as now observed in Frame $D$, is obtained by a similarity transform $D_{DB}D_{12}D_{DB}^{-1}$:

$$\begin{aligned} D_{DB}D_{12}D_{DB}^{-1} &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(-\vec{p}, I)^{-1} & (1) \\ &= (-\vec{p}, I)(\vec{d}_{12}, R_{12})(+\vec{p}, I) & (2) \\ &= (-\vec{p}, I)((\vec{d}_{12} + R_{12}\vec{p}), R_{12}) & (3) \\ &= ((\vec{d}_{12} + (R_{12} - I)\vec{p}), R_{12}) & (4) \end{aligned}$$

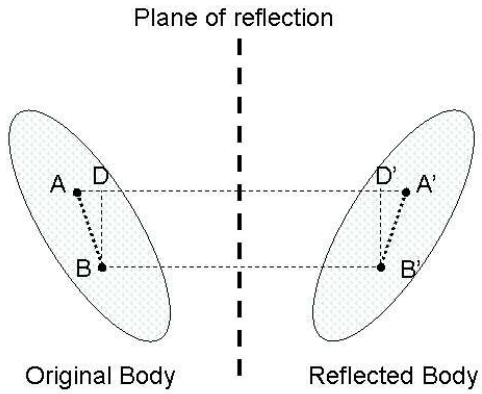


Figure 1: Diagram of reflection process

Hence, if $\vec{p} = -(R_{12} - I)^{-1}\vec{d}_{12} = (I - R_{12})^{-1}\vec{d}_{12}$, then $D_{DB}D_{12}D_{DB}^{-1} = (\vec{0}, R_{12})$. I.e., as viewed in reference Frame $D$, the displacement is a pure rotation by amount $R_{12}$.

**Problem 3:**

**Part (a):** There are many ways that one can prove that reflections preserve length. Here is one approach (see Figure 1).

Select any two non-identical points, $A$ and $B$, in a rigid body. After reflection, those points become $A'$ and $B'$. Form the right triangle $ABD$, where the line $BD$ is chosen to be perpendicular to the line $AA'$. Similary, in the reflected body, form the right triangle $A'B'D'$. Simple geometric arguments show that since the distance $|BD|$ and $|B'D'|$ are equal, and the distances $|AD|$ and $|A'D'|$ are equal, then $|AB| = |A'B'|$. Hence, the distance between $A$ and $B$ is preserved under reflection. Since $A$ and $B$ were chosen randomly, the result will hold for any non-identical pair of points in the body. Thus, distance is always preserved under reflection.

**Part (b):** Generally, physically meaningful planar displacements are not equivalent to a single reflection. To see this, define three points $(A, B, C)$ in the body of Figure 1. Because the body is rigid, one can think of points $(A, B, C)$ as forming a rigid triangle. Consider the triangle formed from the reflected points $(A', B', C')$. Note that it is impossible physically translate $(A, B, C)$ to $(A', B', C')$. Finally, note that any rigid body planar displacement can generally be realized as the result of two sequential reflections.

**Problem 4:** You were to "prove" that a body undergoing spherical motion has three degrees of freedom.

A body undergoing spherical motion has one fixed point. Let the body consist of $N$ particles. Let $P_1$ denote the particle lying at the fixed point. A point in 3-dimensional Euclidean space normally requires 3 independent variables to fix its location. However, since $P_1$ does not move, it actually has 0 degrees-of-freedom (DOF). Now consider a particle $P_2$ in the body. Particle $P_2$ has 3 DOF as a particle. However, it is constrained to lie a fixed distance, $d_{12}$ from particle $P_1$ due to the fact that $P_1$ and $P_2$ are part of the same rigid body. The fixed distance relationship imposes one constraint on $P_2$. Next consider a point $P_3$, which lie a fixed distance from $P_1$ and $P_2$. Therefore, there are two constraints on its location. Now, consider a particle $P_4$. Since its must lie a fixed distance from $P_1$, $P_2$, and $P_3$, there are three constraints on its motion. Particles $P_5, \ldots, P_N$ similarly have 3 constraints.

The total number of degrees of freedom of the $N$ particles are: $3(N-1)+0 = 3N-3$. The total number of constraints on these particles are: $1+2+3(N-3) = 3N-6$. Hence, the total net DOF of a body is the number of freedoms of the particles minus the number of constraints that bind them into a rigid body: $(3N-3)-(3N-6) = 3$.