ME 115(a): Solution to Homework #1

Solution to Problem #1

The key idea is that the pole is fixed in both the moving and observer reference frames. Hence, before displacement, the pole is located at some position $\vec{v}_B$ as seen by an observer in the body fixed frame. After displacement, the observer in the body fixed frame also sees the pole in his/her coordinates at point $\vec{v}_B$. However, the moving body has displaced relative to the fixed observer by amount $D_2 = (R_{12}, \vec{d}_{12})$, so points before and after the displacement are related by the coodinate transform:

$$\vec{v}_B = \vec{d}_{12} + R_{12}\vec{v}_B.$$

This equation can be solved to find the pole location in frames B and C:

$$\vec{v}_B = \vec{v}_C = (I - R_{12})^{-1}\vec{d}_{12}$$

Now points in the body fixed frame (before the displacement) are related to points in the fixed reference frame by $D_1 = (R_{01}, \vec{d}_{01})$. Therefore the location of the pole in the fixed referece frame is:

$$\vec{v}_A = \vec{d}_{01} + R_{01}(I - R_{12})^{-1}\vec{d}_{12}$$

Solution to Problem #2

If $D_1 = (x, y, \theta) = (1.0, 2.0, 30.0^o)$ and $D_2 = (x, y, \theta) = (2.0, 2.0, 45^o)$, then using the results from problem 1 above:

$$\begin{aligned}
\vec{v}_B &= (I - R_{12})^{-1}\vec{d}_{12} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos(45^o) & -\sin(45^o) \\ \sin(45^o) & \cos(45^o) \end{bmatrix} \right)^{-1} \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix} . \\
&= \begin{bmatrix} -1.41 \\ 3.41 \end{bmatrix}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\vec{v}_A &= \vec{d}_{01} + R_{01}\vec{v}_B \\
&= \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} + \begin{bmatrix} \cos(30^o) & -\sin(30^0) \\ \sin(30^o) & \cos(30^o) \end{bmatrix} \begin{bmatrix} -1.41 \\ 3.41 \end{bmatrix} . \\
&= \begin{bmatrix} -1.93 \\ 4.25 \end{bmatrix}
\end{aligned}$$

Solution to Problem #3

Assume the same setup as in Problem 2. Assume that the basis vectors of frame A are parallel to frame B and its origin lies at the pole of the displacement from B to C. The location of frame B relative to frame A is a pure translation of amount $\vec{d}_{01} = -\vec{v}_B = -(I - R_{12})^{-1}\vec{d}_{12}$ The displacement from B to C, as described in Frame A, can be computed as $D_1 D_2 D_1^{-1}$:

$$\begin{aligned}
D_1 D_2 D_1^{-1} &= (I, \vec{d}_{01})(R_{12}, \vec{d}_{12})(I, -\vec{d}_{12}) \\
&= (I, \vec{d}_{01})(R_{12}, \vec{d}_{12} - R_{12}\vec{d}_{01}) \\
&= (R_{12}, \vec{d}_{12} + (I - R_{12})\vec{d}_{01}) \\
&= (R_{12}, \vec{d}_{12} - (I - R_{12})(I - R_{12})^{-1}\vec{d}_{12}) \\
&= (R_{12}, 0)
\end{aligned}$$

The displacement, as measured in a frame whose origin is located at the pole, is a pure rotation.

Solution to Problem #4

Any rational intuitive argument is acceptable.

Solution to Problem #5

Consider a body which is fixed at the point O, but is free to rotate about that point. Now consider some other point in the body, P_1. Point P_1 has three degrees of freedom in space, but one constraint, namely a fixed distance between P_1 and O.

Next consider another point, P_2. Point P_2 also has three degrees of freedom, but it has two constraints, namely fixed distances between P_2 and O, and between P_2 and P_1.

Consider yet another point, P_3. Point P_3 also has three degrees of freedom, but it has three constrants, namely fixed distances between P_3 and O, between P_3 and P_1, and between P_3 and P_2.

Any other point will have three degrees of freedom, and three (independent) constraints. Therefore, the body has $3 + 3 + 3 - (1 + 2 + 3) = 3$ degrees of freedom.