ME 115(a): Solution to Homework #1

Solution to Problem #1

The key idea is that the pole is fixed in both the moving and observor reference frames. Hence, before displacement, the pole is located at some position ${}^0\vec{v}$ as seen by an observor in the fixed frame. After displacement, the observor in the body fixed frame also sees the pole in his/her coordinates at point ${}^0\vec{v}$. However, the moving body has displaced relative to the fixed observer by amount $g_{12} = (\vec{d}_{12}, R_{12})$. But points in the observer and displaced reference frames are related by the coodinate transform:

$$^0\vec{v} = \vec{d}_{12} + R^0_{12}\vec{v}.$$

This equation can be solved to find the pole location:

$$^0\vec{v} = (I - R_{12})^{-1}\vec{d}_{12}$$

Solution to Problem #2

Let ${}^B\mathbf{P}$ be a vector from the origin of Frame B to the pole of the displacement, where ${}^B\mathbf{P}$ is described in the B coordinate frame. Let the vector from the origin of Frame C to the pole be denoted by ${}^C\mathbf{P}$, where ${}^C\mathbf{P}$ is described in Frame C. The description of ${}^C\mathbf{P}$ in Frame B can be computed using the following relation:

$$^B\mathbf{P} = R_{12}\ {}^C\mathbf{P}\ +\ \overline{d}_{12}$$

But, since the pole of the displacement remains fixed in both frames during the displacement, ${}^B\mathbf{P} =^C \mathbf{P}$, and therfore, we can compute ${}^B\mathbf{P}$ by solving the equation:

$$^B\mathbf{P} = R_{12}\ {}^B\mathbf{P}\ +\ \overline{d}_{12}$$

which has solution:

$$^B\mathbf{P} = (I\ -\ R_{12})^{-1}\overline{d}_{12}$$

The location of the pole in Frame A can be determined by a simple transformation.

1. **Description in Frame A**: ${}^A\mathbf{P} = R_{01}\ {}^B\mathbf{P} + \overline{d}_{01} = R_{01}(I - R_{12})^{-1}\overline{d}_{12} + \overline{d}_{01}$
2. **Description in Frame B**: ${}^B\mathbf{P} = (I - R_{12})^{-1}\overline{d}_{12}$.
3. **Description in Frame C**: ${}^C\mathbf{P} =^B \mathbf{P} = (I - R_{12})^{-1}\overline{d}_{12}$.

If $D_1 = (x, y, \theta) = (1.0, 1.0, 45.0^o)$ and $D_2 = (x, y, \theta) = (3.0, 3.0, 30^o)$, then using the formula above:

$$\begin{aligned} {}^B\mathbf{P} &= (I - R_{12})^{-1}\overline{d}_{12} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos(30^o) & -\sin(30^o) \\ \sin(30^o) & \cos(30^o) \end{bmatrix} \right)^{-1} \begin{bmatrix} 3.0 \\ 3.0 \end{bmatrix}. \\ &= \begin{bmatrix} 11.1962 \\ 3.0 \end{bmatrix} \end{aligned}$$

The result can be transformed to any desired coordinate system.

Solution to Problem #3

Assume the same setup as in Problem 2. Select a new Frame, Frame D, whose basis vectors are parallel to Frame B and whose origin lies at the pole of the displacement from B to C. Let D_{13} denote the location of Frame B relative to Frame D. The location of Frame B relative to Frame D is a pure translation of amount $-{}^{B}\mathbf{P}$, and therefore, $D_{13} = (I, (I - R_{12})^{-1}\overline{d}_{12})$. The displacement from B to C, as described in Frame D, can be computed as $D_{13}D_{12}D_{13}^{-1}$:

$$
\begin{aligned}
D_{13}D_{12}D_{13}^{-1} &= (I, -(I - R_{12})^{-1}\overline{d}_{12})(R_{12}, \overline{d}_{12})(I, (I - R_{12})^{-1}\overline{d}_{12}) \\
&= (I, -(I - R_{12})^{-1}\overline{d}_{12})(R_{12}, R_{12}(I - R_{12})^{-1}\overline{d}_{12} + \overline{d}_{12}) \\
&= (R_{12}, R_{12}(I - R_{12})^{-1}\overline{d}_{12} + \overline{d}_{12} - (I - R_{12})^{-1}\overline{d}_{12}) \\
&= (R_{12}, -(I - R_{12})(I - R_{12})^{-1}\overline{d}_{12} + \overline{d}_{12}) = (R_{12}, -\overline{d}_{12} + \overline{d}_{12}) \\
&= (R_{12}, 0)
\end{aligned}
$$

The displacement, as measured in a frame whose origin is located at the pole, is a pure rotation.

Solution to Problem #4: (Problem 3(c) in Chapt. 2 of MLS)

Part (a): Let at 3×3 matrix R have the form:

$$
\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = [\, \vec{r}_1 \quad \vec{r}_2 \quad \vec{r}_3 \,]
$$

Take the determinant of R using cofactors:

$$
\begin{aligned}
det(R) &= r_{11}(r_{22}r_{33} - r_{32}r_{23}) - r_{21}(r_{12}r_{33} - r_{32}r_{13}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}) \\
&= \vec{r}_1^{\,T} \begin{bmatrix} r_{22}r_{33} - r_{32}r_{23} \\ r_{32}r_{13} - r_{12}r_{33} \\ r_{12}r_{23} - r_{22}r_{13} \end{bmatrix} = \vec{r}_1^{\,T}(\vec{r}_2 \times \vec{r}_3)
\end{aligned}
$$