ME 115B: Solution to Homework #1

Problem #1:

$$G_{st}(\theta) = \begin{bmatrix} \mathrm{R}(\theta) & \mathrm{p}(\theta) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1)\cos(\theta_2+\theta_3) & -\sin(\theta_1) & \cos(\theta_1)\sin(\theta_2+\theta_3) & \cos(\theta_1)[l_2\cos(\theta_2)+l_3\cos(\theta_2+\theta_3)] \\ \sin(\theta_1)\cos(\theta_2+\theta_3) & \cos(\theta_1) & \sin(\theta_1)\sin(\theta_2+\theta_3) & \sin(\theta_1)[l_2\cos(\theta_2)+l_3\cos(\theta_2+\theta_3)] \\ -\sin(\theta_2+\theta_3) & 0 & \cos(\theta_2+\theta_3) & -l_2\sin(\theta_2)-l_3\sin(\theta_2+\theta_3) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

We know the hybrid Jacobian is defined as: $\begin{bmatrix} \frac{\partial p}{\partial \theta} \\ (\frac{\partial R}{\partial \theta} * R^{-1})^{\curlyvee} \end{bmatrix}$

$$J^H = \begin{bmatrix} -\sin(\theta_1)[l_2\cos(\theta_2)+l_3\cos(\theta_2+\theta_3)] & -\cos(\theta_1)[l_2\sin(\theta_2)+l_3\sin(\theta_2+\theta_3)] & -l_3\cos(\theta_1)\sin(\theta_2+\theta_3) \\ \cos(\theta_1)[l_2\cos(\theta_2)+l_3\cos(\theta_2+\theta_3)] & -\sin(\theta_1)[l_2\sin(\theta_2)+l_3\sin(\theta_2+\theta_3)] & -l_3\sin(\theta_1)\sin(\theta_2+\theta_3) \\ 0 & -l_2\cos(\theta_2)-l_3\cos(\theta_2+\theta_3) & -l_3\cos(\theta_2+\theta_3) \\ 0 & -\sin(\theta_1) & -\sin(\theta_1) \\ 0 & \cos(\theta_1) & \cos(\theta_1) \\ 1 & 0 & 0 \end{bmatrix}$$

Alternatively, we know the determinant of the Jacobian as:

$$\det\left(\frac{\partial p}{\partial \theta}\right) = -l_2 l_3[l_2\cos(\theta_2)+l_3\cos(\theta_2+\theta_3)]\sin(\theta_3)$$

There exist singularities when the determinant of the Jacobian is zero. This corresponds to two cases. The first one is when $\theta_3 = 0$ and θ_2 can take on any values. This is when the elbow manipulator is stretched out. The other cases is when $\theta_2 = \left(\frac{\pi}{2}\right)$ and $\theta_3 = \pi$.

Problem #2:

The forward kinematic of a two-link planar manipulator is given as:

$x = l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2)$, $y = l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2)$

$$J = \left(\frac{\partial p}{\partial \theta}\right) = \begin{bmatrix} -l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos(\theta_1)+l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \end{bmatrix}$$

There are isotropic points if the ratio of the maximum singular value to the minimum singular value is equal to 1 or the maximum eigenvalue to the minimum eigenvalue is 1. With the former case,

singualar values $\Rightarrow \det(J^2 - \lambda I) = 0$

For the latter case,

eigenvalues $\Rightarrow \det(J - \lambda I) = 0$

In both cases, the values are of the form, $a \pm \sqrt[2]{b}$, so isotropic points occur when b $\longrightarrow 0$.

Case 1:

$$b = l_1^4 + 4l_2^4 + 4l_1l_2(l_1^2+2l_2^2)\cos\theta_2 + 8l_1^2l_2^2\cos^2\theta_2$$
$$\cos\theta_2 = \frac{1}{16l_1^2l_2^2}\left[-4l_1l_2(l_1^2+2l_2^2) \pm (4l_1l_2(l_1^2-2l_2^2))i\right]$$

Since θ_2 cannot be equal to an imaginary number: $l_1^2 - 2l_2^2 = 0$ or $l_1 = \sqrt[2]{2}l_2$ and $\theta_2 = 135°, 225°$.

Case 2:

$$\mathrm{b} = -4l_1l_2(l_1^2+2l_2^2)\sin\theta_2 + (-l_2\cos(\theta_1+\theta_2)+l_1\sin(\theta_1)+l_2\sin(\theta_1+\theta_2))^2 = 0$$

This is true if $l_1 = l_2$ and $\theta_1 = \theta_2 = 90°$.

Problem #3:

$$\mathrm{J} = \begin{bmatrix} -q_1\times\omega_1 & -q_2\times\omega_2 & -q_3\times\omega_3 & -q_4\times\omega_4 & -q_5\times\omega_5 & -q_6\times\omega_6 \\ \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 & \omega_6 \end{bmatrix}$$

Choosing q as the intersection between the axis of rotation and the line, and the z-axis of the frame of reference points along the line then

$$q_n = \begin{bmatrix} 0 \\ 0 \\ q_{nz} \end{bmatrix} \text{ and } q_n\times\omega_n = \begin{bmatrix} -q_{nz}\omega_{ny} \\ q_{nz}\omega_{nx} \\ 0 \end{bmatrix}.$$

So the third row of J is composed of zeros. Rank = dim(row s) = dim(col s).For full rank, the rank would need to be six. But, with one row of zeros, we know the ranks drops, so there exists a singularity.

Problem #4:

a) N = 3 links, g = 4 joints, f_i = 1 DOF/link F = 3(3-4) +5 = 1DOF (Gruebler's formula)

The spatial version of Gruebler's formula cannot be used because the constraints imposed by the joints will not be independent.

b) Structure Equation: $g_{st} = g_{s1}g_{12}g_{23}g_{3t}g_{st}(0) = I$, since this is a closed loop mechanism.

c) $\mathrm{J} = \begin{bmatrix}\xi_1 & \xi_2' & \xi_3' & \xi_4'\end{bmatrix}$ From Chapter 2, the planar twists for revolute and prismatic joints are respectively:

$$\xi_{\mathrm{Re}\,volute} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix} \qquad and \qquad \xi_{\mathrm{Pr}\,ismatic} = \begin{bmatrix} q_x \\ q_y \\ 0 \end{bmatrix}$$

The Jacobian for the structure equation is:

$$\mathrm{J} = \begin{bmatrix} 0 & l_1\sin(\theta_1) & 0 & 1 \\ 0 & -l_1\cos(\theta_1) & -l_1\cos(\theta_1) - l_2\cos(\theta_2-\theta_1) & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

d) Since $\mathrm{J}\theta' = 0$ because the forward kinematic is equal to the identity or a constant, if d is the active variable then:

$$\xi_4\theta' = -\begin{bmatrix}\xi_3' & \xi_2' & \xi_1\end{bmatrix}\alpha'$$

Let $\mathrm{J}_{append} = \begin{bmatrix}\xi_3' & \xi_2' & \xi_1\end{bmatrix}$ so $\det(J_{append}^T J_{append}) = [l_1\sin(\theta_1)(l_1\cos(\theta_1)+l_2\cos(\theta_1+\theta_2))]$

Singularities exist if:

Case 1: $\sin(\theta_1) = 0 \Longrightarrow \theta_1 = 0, \theta_2 = 0$ (Mechanism is flat.)

Case 2: $l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) = 0 \Longrightarrow \theta_1 = \frac{\pi}{2}, \theta_2 = \pi$ and $l_1 = l_2$

e) If θ_1 is the active variable, then $\xi_1 \theta' = -\begin{bmatrix} \xi_2' & \xi_3' & \xi_4' \end{bmatrix} \alpha', J_{append} = \begin{bmatrix} \xi_2' & \xi_3' & \xi_4' \end{bmatrix}$

$\det(J_{append}^T J_{append}) = l_1^2 l_2^2 \cos^2(\theta_1 - \theta_2) \Longrightarrow \cos(\theta_1 - \theta_2) = 0 \Longrightarrow \theta_1 - \theta_2 = \frac{\pi}{2}$

So singularity exists if $\theta_1 - \theta_2 = \frac{\pi}{2}, l_1 \geqq l_2$.