ME 115(a): Solution to Homework #2

Problem 1:

- **(a)** (problem 3(c) in chapter 2 of the MLS text). Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix} .$$

Expanding $det(R)$ using cofactors, one finds:

$$\begin{aligned} det(R) &= r_{11}(r_{22}r_{33} - r_{32}r_{23}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}) \\ &= \vec{r}_1^T \cdot (\vec{r}_2 \times \vec{r}_3) \end{aligned}$$

- **(b)** To show that $\mathrm{cofactor(r_{ii}) = r_{ii}}$, let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} .$$

The columns of this matrix are unit vectors, which can be denoted as:

$$\mathbf{x} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

These columns can be interpreted as the unit vectors of an orthogonal right handed coordinate system. Consequently,

$$\mathbf{x} = \mathbf{y} \times \mathbf{z}; \qquad \mathbf{y} = \mathbf{z} \times \mathbf{x}; \qquad \mathbf{z} = \mathbf{x} \times \mathbf{y}$$

Performing the cross product and equating sides for $\mathbf{x} = \mathbf{y} \times \mathbf{z}$, we get the relation:

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} = \begin{bmatrix} r_{22}r_{33} - r_{23}r_{32} \\ r_{13}r_{32} - r_{12}r_{33} \\ r_{12}r_{23} - r_{13}r_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{cofactor(r_{11})} \\ \mathrm{cofactor(r_{21})} \\ \mathrm{cofactor(r_{31})} \end{bmatrix} \tag{1}$$

Similar relationships can be derived for the other columns to show that $r_{ij} = \mathrm{cofactor}(r_{ij})$ for all elements of a special orthogonal matrix.

Problem 2: (Problem 5(c) of chapter 2 in the MLS text). There are two ways to solve this. The simplest way is to use the result of part 5(b) quoted in the text:

$$R = \frac{1}{1+||a||^2}\begin{bmatrix} 1+a_1^2-a_2^2-a_3^2 & 2(a_1a_2-a_3) & 2(a_1a_3+a_2) \\ 2(a_1a_2+a_3) & 1-a_1^2+a_2^2-a_3^2 & 2(a_2a_3-a_1) \\ 2(a_1a_3-a_2) & 2(a_2a_3+a_1) & 1-a_1^2-a_2^2+a_3^2 \end{bmatrix} \quad (2)$$

where $||a||^2$ is shorthand notation for $||a||^2 = a_1^2 + a_2^2 + a_3^2$. Noting that

$$trace(R) = \frac{3-||a||^2}{1+||a||^2} \Rightarrow ||a||^2 = \frac{3-trace(R)}{1+trace(R)} = \frac{3-r_{11}-r_{22}-r_{33}}{1+r_{11}+r_{22}+r_{33}}$$

so that an expression for $||a||^2$ is known, simple algebraic manipulation of the off-diagonal term of R yield

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1+||a||^2}{4}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$$

If you didn't use the results of 5(b) in the text, then you would have started with Cayley's formula $R = (I-\hat{a})^{-1}(I+\hat{a})$ and derived Equation (2).

Problem 3: Let Z-X-Y Euler angles be denoted by ψ, ϕ, and γ.

- **Part (a):** To develop an expression for the Z-Y-X rotation, first find expressions for each of the individual rotation matrices:

$$R_\psi = Rot(\vec{z},\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_\phi = Rot(\vec{y},\phi) = \begin{bmatrix} \cos\phi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$R_\gamma = Rot(\vec{x},\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

where the notation $R(\vec{p},\alpha)$ means rotation by angle α about axis $\vec{p}$. The final expression is obtained by multiplying these matrices:

$$R(\psi,\phi,\gamma) = R_\psi R_\phi R_\gamma = \begin{bmatrix} c\psi c\phi & c\psi s\phi s\gamma - s\psi c\gamma & c\psi s\phi c\gamma + s\psi s\gamma \\ s\psi c\phi & s\psi s\phi s\gamma + c\psi c\gamma & s\psi s\phi c\gamma - c\psi s\gamma \\ -s\phi & c\phi s\gamma & c\phi c\gamma \end{bmatrix}$$

where $s\psi = \sin\psi$, $c\phi = \cos\phi$, etc.

- **Part (b):** To find an expression for the angles ψ, ϕ, and γ as a function of the r_{ij} in the matrix

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

note that $\sin\phi = -r_{31}$. There are two solutions to this equation: $\phi_1 = \sin^{-1}(r_{31})$ and $\phi_2 = \pi - \phi_1$. Manipulation of the (r_{11}, r_{21}) and (r_{32}, r_{33}) terms yields:

$$\psi = Atan2[\frac{r_{21}}{\cos\phi}, \frac{r_{11}}{\cos\phi}]; \qquad \gamma = Atan2[\frac{r_{32}}{\cos\phi}, \frac{r_{33}}{\cos\phi}].$$

Problem 4: (Problem 4(a,b) in Chapter 2 of MLS).

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}
(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
&= R(\vec{w} \times (R^T\vec{v})) \\
&= (R\vec{w}) \times (RR^T\vec{v}) \\
&= (R\vec{w}) \times \vec{v} \\
&= (\widehat{R\vec{w}})\vec{v}
\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}
(R\vec{v}) \times (R\vec{w}) &= (\widehat{R\vec{v}})(R\vec{w}) \\
&= (R\hat{v}R^T)(R\vec{w}) \\
&= R\hat{v}R^TR\vec{w} \\
&= R(\hat{v}\vec{w}) \\
&= R(\vec{v} \times \vec{w})
\end{aligned}$$

Problem 5: (Problem 8(b) in Chapter 2 of MLS).

$$\begin{aligned}
e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\
&= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\
&= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\
&= ge^{\Lambda}g^{-1}
\end{aligned}$$