ME 115(a): Solution to Homework #2

Problem 1:

- **(a)** (problem 3(c) in chapter 2 of the MLS text). Let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix}.$$

  Expanding $det(R)$ using cofactors, one finds:

$$\begin{aligned} det(R) &= r_{11}(r_{22}r_{33} - r_{32}r_{23}) + r_{21}(r_{32}r_{13} - r_{12}r_{33}) + r_{31}(r_{12}r_{23} - r_{22}r_{13}) \\ &= \vec{r}_1^{\,T} \cdot (\vec{r}_2 \times \vec{r}_3) \end{aligned}$$

- **(b)** To show that $\mathrm{cofactor(r_{ii})} = \mathrm{r_{ii}}$, let

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

  The columns of this matrix are unit vectors, which can be denoted as:

$$\mathbf{x} = \begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix}$$

  These columns can be interpreted as the unit vectors of an orthogonal right handed coordinate system. Consequently,

$$\mathbf{x} = \mathbf{y} \times \mathbf{z}; \qquad \mathbf{y} = \mathbf{z} \times \mathbf{x}; \qquad \mathbf{z} = \mathbf{x} \times \mathbf{y}$$

  Performing the cross product and equating sides for $\mathbf{x} = \mathbf{y} \times \mathbf{z}$, we get the relation:

$$\begin{bmatrix} r_{11} \\ r_{21} \\ r_{31} \end{bmatrix} = \begin{bmatrix} r_{22}r_{33} - r_{23}r_{32} \\ r_{13}r_{32} - r_{12}r_{33} \\ r_{12}r_{23} - r_{13}r_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{cofactor(r_{11})} \\ \mathrm{cofactor(r_{21})} \\ \mathrm{cofactor(r_{31})} \end{bmatrix} \tag{1}$$

  Similar relationships can be derived for the other columns to show that $r_{ij} = \mathrm{cofactor}(r_{ij})$ for all elements of a special orthogonal matrix.

Problem 2: (Problem 4(a,b) in Chapter 2 of MLS).

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\ &= R(\vec{w} \times (R^T\vec{v})) \\ &= (R\vec{w}) \times (RR^T\vec{v}) \\ &= (R\vec{w}) \times \vec{v} \\ &= (\widehat{R\vec{w}})\vec{v}\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}(R\vec{v}) \times (R\vec{w}) &= (\widehat{R\vec{v}})(R\vec{w}) \\ &= (R\hat{v}R^T)(R\vec{w}) \\ &= R\hat{v}R^TR\vec{w} \\ &= R(\hat{v}\vec{w}) \\ &= R(\vec{v} \times \vec{w})\end{aligned}$$

Problem 3: Can every orthogonal matrix be represented by a the exponential of a real matrix?

You should have either remembered or derived the fact that $det(e^C) = e^{tr(C)}$, where $tr(C)$ is the trace of determinant C. Note that if $tr(C)$ is real, than $e^{tr(C)}$ is always a positive number and therefore orthogonal matrices with determinant -1 can not be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant. But, the two subsets are disjoint.

Note, that if $tr(C) = \frac{\pi}{2}i$ (where $i^2 = -1$), then $det(e^C) = -1$. However, this can not be true if C is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let C be a $n \times n$ matrix. If n is even, then all of the eigenvalues of C must be complex cojugates and or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if n is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum of the eigenvalues must also be real number. Thus, if C is a real matrix (as specified in part (a)), then e^C can not represent orthogonal matrices with determinant -1.

Problem 4: Find the axis of rotation and angle of rotation associated with the rotation matrix:

Problem 4: Consider the following rotation matrix:

$$\begin{bmatrix} 0.882772 & -0.416266 & 0.217798 \\ 0.44976 & 0.882772 & -0.135756 \\ -0.135756 & 0.217798 & 0.966506 \end{bmatrix}$$

From Eq. (2.17) in the MLS text:

$$\cos(\phi) = \frac{r_{11} + r_{22} + r_{33} - 1}{2} = \frac{0.882772 + 0.882772 + 0.966505 - 1.0}{2} = 0.8660245$$

Thus, $\phi = \cos^{-1}(0.8660245) = 30^o$. Thus, $\sin(\phi) = 0.5$, and therefore from Eq. (2.18) of the MLS text:

$$\begin{aligned} \omega_x &= \tfrac{r_{32} - r_{23}}{2\sin\phi} = 0.353554 \\ \omega_y &= \tfrac{r_{13} - r_{31}}{2\sin\phi} = 0.353554 \\ \omega_z &= \tfrac{r_{21} - r_{12}}{2\sin\phi} = 0.866026 \end{aligned}$$

Problem 5: (Problem 8(b) in Chapter 2 of MLS).

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\ &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\ &= ge^{\Lambda}g^{-1} \end{aligned}$$