ME 115(a): Solution #2

Solution to Problem #1

Part (a): Let R_z, R_y, and R_x be the rotation matrices for each axis. Then,

$$R_z = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$$R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{bmatrix}$$

Then $R = R_z R_y R_x$:

$$R = \begin{bmatrix} \cos(\psi)\cos(\phi) & -\sin(\psi)\cos(\gamma) + \cos(\psi)\sin(\phi)\sin(\gamma) & \sin(\psi)\sin(\gamma) + \cos(\psi)\sin(\phi)\cos(\gamma) \\ \sin(\psi)\cos(\phi) & \cos(\psi)\cos(\gamma) + \sin(\psi)\sin(\phi)\sin(\gamma) & -\cos(\psi)\sin(\gamma) + \sin(\psi)\sin(\phi)\cos(\gamma) \\ -\sin(\phi) & \cos(\phi)\sin(\gamma) & \cos(\phi)\cos(\gamma) \end{bmatrix}$$

Part (b):

Given the rotation matrix:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

We can equate terms with the expression from part (a). The expression for $r_3 1$ gives us an equation to find ϕ:

$$\phi = \sin^{-1}(-r_{31})$$

There are two distinct values of ϕ which will satisfy this equation. Choosing one of these two values, and using terms r_{21} and r_{11} gives us an expression for ψ:

$$\psi = Atan2\left(\frac{r_{21}}{\cos(\phi)}, \frac{r_{11}}{\cos(\phi)}\right)$$

Similarly, terms r_{32} and r_{33} give an expression for γ:

$$\gamma = Atan2\left(\frac{r_{32}}{\cos(\phi)}, \frac{r_{33}}{\cos(\phi)}\right)$$

Solution Problem #2:

Part (a): The rotation matrix is:

$$\begin{bmatrix} 0.882772 & -0.416266 & 0.217798 \\ 0.44976 & 0.882772 & -0.135756 \\ -0.135756 & 0.217798 & 0.966506 \end{bmatrix}$$

From class notes or the text:

$$\cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.882772 + 0.882772 + 0.966505 - 1.0}{2} = 0.8660245$$

Thus, $\phi = \cos^{-1}(0.8660245) = 30^o$. Thus, $\sin(\phi) = 0.5$, and therefore:

$$\begin{aligned} \omega_x &= \tfrac{a_{32} - a_{23}}{2\sin\phi} = 0.353554 \\ \omega_y &= \tfrac{a_{13} - a_{31}}{2\sin\phi} = 0.353554 \\ \omega_z &= \tfrac{a_{21} - a_{12}}{2\sin\phi} = 0.866026 \end{aligned}$$

Part (b): Similarly, using the relatioships found in problem 1 part(a):

$$\begin{aligned} \phi &= \sin^{-1}(-r_{31}) = \sin^{-1}(0.135756) = 0.136176rad = 7.80234deg \\ \psi &= Atan2\left(\tfrac{r_{21}}{\cos(\phi)}, \tfrac{r_{11}}{\cos(\phi)}\right) = Atan2\left(8.71278, 17.1011\right) = 0.471207rad = 26.9982deg \\ \gamma &= Atan2\left(\tfrac{r_{32}}{\cos(\phi)}, \tfrac{r_{33}}{\cos(\phi)}\right) = Atan2\left(4.21920, 18.7232\right) = 0.221643rad = 12.6992deg \end{aligned}$$

Solution to Problem #3 (Prob. 4(a,b) in Chapt 2 of MLS)

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$
\begin{aligned}
(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
&= R(\vec{w} \times (R^T\vec{v})) \\
&= (R\vec{w}) \times (RR^T\vec{v}) \\
&= (R\vec{w}) \times \vec{v} \\
&= (\widehat{R\vec{w}})\vec{v}
\end{aligned}
$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$
\begin{aligned}
(R\vec{v}) \times (R\vec{w}) &= (\widehat{R\vec{v}})(R\vec{w}) \\
&= (R\hat{v}R^T)(R\vec{w}) \\
&= R\hat{v}R^T R\vec{w} \\
&= R(\hat{v}\vec{w}) \\
&= R(\vec{v} \times \vec{w})
\end{aligned}
$$

Solution to Problem #4 (Prob. 6(a,b,d) in Chapt 2 of MLS)

Part (a):

Since it is shown in the text that the set of all quaternians satisfies the axioms of a group, it is only slightly more to show that the set of **unit** quaternians also satisfies the axioms.

1. Closure: We must show that for any two unit quaternians Q and P, the product $Q \cdot P$ is also a unit quaternian. Let R be the product $Q \cdot P$. Then, $R = (r_0, \vec{r}) = (q_0p_0 - \vec{q} \cdot \vec{p}, q_0\vec{p} + p_0\vec{q} + \vec{q} \times \vec{p})$. The magnitude of the product is:

$$
\begin{aligned}
||R||^2 &= r_0^2 + \vec{r} \cdot \vec{r} \\
&= (p_3q_0 + p_2q_1 - p_1q_2 + p_0q_3)^2 + (p_2q_0 - p_3q_1 + p_0q_2 + p_1q_3)^2 + \\
&\quad (p_1q_0 + p_0q_1 + p_3q_2 - p_2q_3)^2 + (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)^2 \\
&= (p_0^2 + p_1^2 + p_2^2 + p_3^2)(q_0^2 + q_1^2 + q_2^2 + q_3^2) \\
&= (\vec{p} \cdot \vec{p})(\vec{q} \cdot \vec{q}) \\
&= 1 * 1 = 1
\end{aligned}
$$

   So, the closure axiom is satisfied by unit quaternians.

2. Associativity: Since the binary product is associative for all quaternians, it must be associative for unit quaternians.

3. Unique Identity: The identity element for all quaternians is $Q = (1, 0)$, which is obviously also a unit quaternian.

4. Unique Inverse: The inverse for a quaternian is given by $Q^{-1} = Q^*/||Q||^2$. For unit quaternians, $||Q||^2 = 1$, so this becomes $Q^{-1} = Q^*$, which is also a unit quaternian, since $Q^* = (q_0, -\vec{q})$, and $\vec{q} \cdot \vec{q} = (-\vec{q}) \cdot (-\vec{q})$.

Part (b): If a unit quaternion, q, has real part q_R and pure part q_P, and $\vec{x} = [x_1\ x_2\ x_3]^T$ is represented as a pure quaternion $\tilde{x} = (0, x_1, x_2, x_3) = 0 + \vec{x}$, then:

$$\begin{aligned} \tilde{x}q^{-1} &= \vec{x} \cdot q_P \qquad (\leftarrow \text{real part}) \\ &+ \ q_R\vec{x} - (\vec{x} \times q_P \qquad (\leftarrow \text{pure part}) \end{aligned}$$

Similarly, the product $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q\tilde{x}q^{-1} &= q_R(\vec{x} \cdot q_P) - q_P \cdot (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{real part}) \\ &+ \ q_R(q_R\vec{x} - \vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{pure part}) \end{aligned}$$

The real part of $q\tilde{x}q^{-1}$ is:

$$(\vec{x} \cdot q_P)q_R - q_P \cdot [q_R\vec{x} - (\vec{x} \times q_P)] = q_R(\vec{x} \cdot q_P) - q_R(\vec{x} \cdot q_P) + q_P \cdot (\vec{x} \times q_P) = 0$$

Thus $q\tilde{x}q^{-1}$ is a pure quaternion when $\tilde{x}$ is.

The vector part of $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q_R(q_R\vec{x} - \vec{x} \times q_P) &+ \ (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \\ &= \ {q_R}^2\vec{x} - q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x}) - q_P \times (\vec{x} \times q_P) \\ &= \ q_R^2\vec{x} - 2q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P - [(q_P \cdot q_P)\vec{x} - (\vec{x} \cdot q_P)q_P] \\ &= \ [q_R^2 - (q_P \cdot q_P)]\vec{x} + 2[(\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x})] \end{aligned}$$

where we have used the triple cross product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Part (d):

(i) If $A_1, A_2 \in SO(3)$, then each of the 9 elements in the product matrix $A_1\ A_2$ requires 3 multiplications and 2 additions. Hence, the product $A_1\ A_2$ requires a total of 27 multiplications and 18 additions.

(ii) Let q_1 and q_2 be quaternions, with respective real and vector parts q_{1R}, q_{2R} and $\vec{q}_{1P}$, $\vec{q}_{2P}$. The real part of the quaternion product, $q_{1R}q_{2R} - \vec{q}_{1P} \cdot \vec{q}_{2P}$, requires 4 multiplications and 3 additions (where the subtraction is counted as an addition). The pure part, $\vec{q}_{3P} = q_{1R}\vec{q}_{2P} + q_{2R}\vec{q}_{1P} + \vec{q}_{1P} \times \vec{q}_{2P}$, can be evaluated in 12 multiplications and 9 additions. Thus, the quaternion product requires a total of 16 multiplications and 12 additions. It is therefore more efficient than the equivalent matrix multiplication.

(iii) The rotation of a vector by multiplication of a 3×3 rotation matrix times a 3×1 vector requires only 9 multiplications and 6 additions.

(iv) The number of multiplications and additions for the equivalent quaternion operation will depend upon the form which one uses for the quaternion vector rotation. Using the identity $1 = q_R^2 + q_P \cdot q_P$, it is possible to show that the vector part of $q\tilde{x}q^{-1}$ in part (b) above can be rearranged to the form:

$$\vec{x} + 2[q_P \times (q_P \times \vec{x}) + q_R(q_P \times \vec{x})]$$

Since $q_P \times \vec{x}$ need only be evaluated once, this takes only 18 multiplications and 12 additions. However, no matter what form one tries, the quaternion approach will always take more operations than the matrix/vector approach for vector rotation.

Solution to Problem #5 (Prob. 8(b) in Chapt 2 of MLS)

$$\begin{aligned} e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\ &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\ &= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\ &= ge^{\Lambda}g^{-1} \end{aligned}$$