ME 115(a): Solution #2

Solution to Problem #1

You should have either remembered or derived the fact that $det(e^C) = e^{tr(C)}$, where $tr(C)$ is the trace of determinant C. Note that if $tr(C)$ is real, than $e^{tr(C)}$ is always a positive number and therefore orthogonal matrices with determinant -1 can not be represented as a matrix exponential. This arises from the fact that the Orthogonal Group is a disconnected group. That is, the matrices with +1 determinant are all connected to each other, and similarly the ones with -1 determinant. But, the two subsets are disjoint.

Note, that if $tr(C) = \frac{\pi}{2} i$ (where $i^2 = -1$), then $det(e^C) = -1$. However, this can not be true if C is real. Recall that the trace of a matrix is equal to the sum of its eigenvalues. Let C be a $n \times n$ matrix. If n is even, then all of the eigenvalues of C must be complex cojugates and or an even number of real roots. Thus, the sum of the eigenvalues must be real. Similarly, if n is odd, the eigenvalues will either be complex conjugates and/or an odd number of real eigenvalues. Thus, the sum of the eigenvalues must also be real number. Thus, if C is a real matrix (as specified in part (a)), then e^C can not represent orthogonal matrices with determinant -1.

Solution Problem #2:

Part (a): The rotation matrix is:

$$\begin{bmatrix} 0.882772 & -0.416266 & 0.217798 \\ 0.44976 & 0.882772 & -0.135756 \\ -0.135756 & 0.217798 & 0.966506 \end{bmatrix}$$

From class notes or the text:

$$\cos(\phi) = \frac{a_{11} + a_{22} + a_{33} - 1}{2} = \frac{0.882772 + 0.882772 + 0.966505 - 1.0}{2} = 0.8660245$$

Thus, $\phi = \cos^{-1}(0.8660245) = 30^o$. Thus, $\sin(\phi) = 0.5$, and therefore:

$$\begin{aligned} s_x &= \tfrac{a_{32} - a_{23}}{2\sin\phi} = 0.353554 \\ s_y &= \tfrac{a_{13} - a_{31}}{2\sin\phi} = 0.353554 \\ s_z &= \tfrac{a_{21} - a_{12}}{2\sin\phi} = 0.866026 \end{aligned}$$

Part (b): Similarly, from class notes you could derive the following relationship between the z-y-z euler angles and the elements of A:

$$\begin{aligned}
\phi &= \cos^{-1}(a_{33}) = \cos^{-1}(0.966506) = 0.259548 rad = 14.871 degrees \\
\gamma &= \text{atan2}(\tfrac{a_{32}}{\sin\phi}, \tfrac{-a_{31}}{\sin\phi}) = \text{atan2}(0.848639, 0.528966) = 0.557384 rad = 31.936 deg \\
\psi &= \text{atan2}(\tfrac{a_{23}}{\sin\phi}, \tfrac{a_{13}}{\sin\phi}) = \text{atan2}(-0.528966, 0.848639) = 2.12818 rad = 121.936 deg
\end{aligned}$$

Solution to Problem #3

Recall that the net rotation of a body in Z-Y-Z Euler Angles is

$$\begin{aligned}
R(\psi,\phi,\gamma) &= Rot(\psi,\vec{z})\ Rot(\phi,\vec{y})\ Rot(\gamma,\vec{z}) \\
&= \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (\cos\psi\cos\phi\cos\gamma - \sin\psi\sin\gamma) & -(\cos\psi\cos\phi\sin\gamma + \sin\psi\cos\gamma) & \cos\psi\sin\phi \\ (\sin\psi\cos\phi\cos\gamma + \cos\psi\sin\gamma) & (-\sin\psi\cos\phi\sin\gamma + \cos\psi\cos\gamma) & \sin\psi\sin\phi \\ -\sin\phi\cos\gamma & \sin\phi\sin\gamma & \cos\phi \end{bmatrix}
\end{aligned}$$

where $Rot(\alpha,\vec{v})$ denotes rotation by angle α about axis $\vec{v}$. Notice that the angle ϕ can be determined from r_{33}:

$$\cos\phi = r_{33} \quad \Rightarrow \quad \phi = \cos^{-1}(r_{33})$$

There will be two solutions to this relationship, which we will indicate by ϕ and $\phi' = -\phi$. The angle γ can be determined from the r_{31} and r_{32} terms:

$$\gamma = Atan2\left(\frac{r_{32}}{\sin\phi}, \frac{-r_{31}}{\sin\phi}\right)$$

for each of the ϕ solutions. The symbol "$Atan2(\cdot,\cdot)$" is the 4-quadrant inverse tangent function.

Similarly, the angle ψ can be computed from the r_{13} and r_{23} terms as:

$$\gamma = Atan2\left(\frac{r_{23}}{\sin\phi}, \frac{r_{13}}{\sin\phi}\right)$$

Solution to Problem #4 (Prob. 4(a,b) in Chapt 2 of MLS)

Part (a): Let's assume that the statement in part (b) of the problem is true. Let $\vec{w}$ be a 3×1 vector and let $\vec{v}$ be any 3×1 vector. Then:

$$\begin{aligned}
(R\hat{w}R^T)\vec{v} &= R\hat{w}(R^T\vec{v}) \\
&= R(\vec{w}\times(R^T\vec{v})) \\
&= (R\vec{w})\times(RR^T\vec{v}) \\
&= (R\vec{w})\times\vec{v} \\
&= (\widehat{R\vec{w}})\vec{v}
\end{aligned}$$

Since this must be true for any vector $\vec{v}$, then $R\hat{w}R^T = (R\vec{w})\hat{}$.

Part (b): We can now assume that part (a) holds.

$$\begin{aligned}
(R\vec{v}) \times (R\vec{w}) &= (\widehat{R\vec{v}})(R\vec{w}) \\
&= (R\hat{v}R^T)(R\vec{w}) \\
&= R\hat{v}R^T R\vec{w} \\
&= R(\hat{v}\vec{w}) \\
&= R(\vec{v} \times \vec{w})
\end{aligned}$$

Solution to Problem #5 (Prob. 8(b) in Chapt 2 of MLS)

$$\begin{aligned}
e^{g\Lambda g^{-1}} &= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda g^{-1})^2 + \frac{1}{3!}(g\Lambda g^{-1})^3 + \cdots \\
&= I + \frac{1}{1!}g\Lambda g^{-1} + \frac{1}{2!}(g\Lambda^2 g^{-1}) + \frac{1}{3!}(g\Lambda^3 g^{-1}) + \cdots \\
&= g(I + \frac{1}{1!}\Lambda + \frac{1}{2!}\Lambda^2 + \frac{1}{3!}\Lambda^3 + \cdots)g^{-1} \\
&= g e^{\Lambda} g^{-1}
\end{aligned}$$