ME 115(b): Solution to Problem Set #2

Problem #1: Slider-Crank linkage.

Derive twist coordinates for the joint axis screws:

We place the origin of our stationary frame at the first joint axis, with the x-axis aligned with the prismatic joint. Then, using the planar rigid body transformations, as described in chapter 2 problem 11 of the MLS text, we can easily see that the twists are given by:

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xi_2 = \begin{bmatrix} l_1 \sin(\theta_1) \\ -l_1 \cos(\theta_1) \\ 1 \end{bmatrix} \xi_3 = \begin{bmatrix} 0 \\ -l_1 \cos(\theta_1) - l_2 \cos(\theta_1 + \theta_2) \\ 1 \end{bmatrix} \xi_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Part(a): First, we get an expression for the grammian matrix. Evaluating all the dot products, it becomes:

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & l_1^2 + 1 & l_1^2 \cos^2(\theta_1) + l_1 l_2 \cos(\theta_1)\cos(\theta_1 + \theta_2) + 1 & l_1 \sin(\theta_1) \\ 1 & l_1^2 \cos^2(\theta_1) + l_1 l_2 \cos(\theta_1)\cos(\theta_1 + \theta_2) + 1 & (l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2))^2 + 1 & 0 \\ 0 & l_1 \sin(\theta_1) & 0 & 1 \end{bmatrix}$$

We are to show that joint 4 has a stationary configuration when $\theta_1 = \theta_2 = 0$. IE, the determinant of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & l_1^2 + 1 & l_1^2 + l_1 l_2 + 1 \\ 1 & l_1^2 + l_1 l_2 + 1 & (l_1 + l_2)^2 + 1 \end{bmatrix}$$

is equal to zero, which is a straightforward calculation.

Part(b): To find the stationary configurations of joint 1, we find the conditions under which the cofactor of the (1,1) element is equal to zero. Simplifying the equation gives the expression

$$\cos(\theta_1 + \theta_2) = 0$$

which implies $\theta_1 + \theta_2 = \pm 90^o$. This configuration corresponds to link 2 being perpendicular to the prismatic joint, or parallel to our y-axis. Therefore, we have the condition $\frac{l_2}{l_1} = \sin(\theta_1)$,

which can only hold if $l_2 < l_1$, so that is our necessary condition for joint 1 to have a stationary configuration.

Problem #2: Problem 14(a) in Chapter 3 of the MLS text

We hope to find the singularities of ZYZ Euler angles. This will happen when the Jacobian of the forward kinematics loses rank. The twists for the "virtual" joints are given by:

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Using the formula for the spatial jacobian, we get

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\sin\alpha & \cos\alpha\sin\beta \\ 0 & \cos\alpha & \sin\alpha\sin\beta \\ 1 & 0 & \cos\beta \end{bmatrix}$$

J loses rank where $det(J^T J) = 0$, or

$$det \begin{bmatrix} 1 & 0 & \cos\beta \\ 0 & 1 & 0 \\ \cos\beta & 0 & 1 \end{bmatrix} = 0$$

or $\cos^2\beta = 1$. Thus, the ZYZ Euler angle parameterization has a singularity at $\beta = 0^o, 180^o$.

Problem #3: Problem 21 in Chapter 3 of the MLS text.

Part(a): Using Gruebler's formula for a planar linkage (as stated in the book), we have

$$F = 3N - \sum (3 - f_i) = 3(N - g) + \sum f_i$$

In our case, $N = 7$, $g = 9$, and all of the $f_i = 1$, so $F = 3(7 - 9) + 9 = 3$.

Part(b): The structure equations are of the form:

$$g_{st} = e^{\hat{\xi}_{11}\theta_{11}} e^{\hat{\xi}_{12}\theta_{12}} e^{\hat{\xi}_{13}\theta_{13}} g_{st}(0) \quad (1)$$

$$= e^{\hat{\xi}_{21}\theta_{21}} e^{\hat{\xi}_{22}\theta_{22}} e^{\hat{\xi}_{23}\theta_{23}} g_{st}(0) \quad (2)$$

$$= e^{\hat{\xi}_{31}\theta_{31}} e^{\hat{\xi}_{32}\theta_{32}} e^{\hat{\xi}_{33}\theta_{33}} g_{st}(0) \quad (3)$$

Again using planar notation, and placing the stationary and tool frames where they appear in the figure in the book, we have:

$$\xi_{11} = \begin{bmatrix} \frac{h}{2} \\ 0 \\ 1 \end{bmatrix}; \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \xi_{13} = \begin{bmatrix} \frac{h}{2} \\ -w \\ 1 \end{bmatrix}$$

$$\xi_{21} = \begin{bmatrix} -\frac{h}{2} \\ 0 \\ 1 \end{bmatrix}; \xi_{22} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \xi_{23} = \begin{bmatrix} -\frac{h}{2} \\ -w \\ 1 \end{bmatrix}$$

$$\xi_{31} = \begin{bmatrix} -\frac{h}{2} \\ 0 \\ 1 \end{bmatrix}; \xi_{32} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix}; \xi_{33} = \begin{bmatrix} \frac{h}{2} \\ -w \\ 1 \end{bmatrix}$$

where $\alpha = \tan^{-1}(\frac{h}{w})$. Also note that the zero configuration of the tool frame is given by

$$g_{st}(0) = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the structure equations are just the simplification of the above expressions.

Part(c): The structure equations can be solved explicitly for d_1, d_2, and d_3, or they can be found geometrically. From the geometry, we can find the position of the endpoints of each link in terms of $g_{st} = ([x, y], R_\phi)$, which leads to an expression for the length of that link.

d_1 connects the points $(0, \frac{h}{2})$ and $(x - \frac{h}{2}\sin\phi, y + \frac{h}{2}\cos\phi)$, which leads to the equation

$$d_1^2 = (x - \frac{h}{2}\sin\phi)^2 + (y + \frac{h}{2}\cos\phi - \frac{h}{2})^2$$

or,

$$d_1 = ((x - \frac{h}{2}\sin\phi)^2 + (y + \frac{h}{2}\cos\phi - \frac{h}{2})^2)^{\frac{1}{2}}$$

Similarly for d_2 and d_3 gives

$$d_2 = ((x + \frac{h}{2}\sin\phi)^2 + (y - \frac{h}{2}\cos\phi + \frac{h}{2})^2)^{\frac{1}{2}}$$

$$d_3 = ((x - \frac{h}{2}\sin\phi)^2 + (y + \frac{h}{2}\cos\phi + \frac{h}{2})^2)^{\frac{1}{2}}$$

Part(d): Since there are three structure equations, there will also be three spatial Jacobians. Each is found in the same way, which is to say

$$J_i = [\,\xi_{i1} \quad \xi'_{i2} \quad \xi'_{i3}\,]$$

Using the twists given above, we can find the jacobians,

$$J_1 = \begin{bmatrix} \frac{h}{2} & \cos\theta_{11} & \frac{h}{2} + (\theta_{12} + w)\sin\theta_{11} \\ 0 & \sin\theta_{11} & -(\theta_{12} + w)\cos\theta_{11} \\ 1 & 0 & 1 \end{bmatrix}$$

$$J_2 = \begin{bmatrix} -\frac{h}{2} & \cos\theta_{21} & -\frac{h}{2} + (\theta_{22} + w)\sin\theta_{21} \\ 0 & \sin\theta_{21} & -(\theta_{22} + w)\cos\theta_{21} \\ 1 & 0 & 1 \end{bmatrix}$$

$$J_3 = \begin{bmatrix} -\frac{h}{2} & \frac{w\cos\theta_{31} - h\sin\theta_{31}}{\sqrt{h^2+w^2}} & -\frac{h}{2} + h(1 + \frac{\theta_{32}}{\sqrt{h^2+w^2}})\cos\theta_{31} + w(1 + \frac{\theta_{32}}{\sqrt{h^2+w^2}}\sin\theta_{31} \\ 0 & \frac{h\cos\theta_{31} + w\sin\theta_{31}}{\sqrt{h^2+w^2}} & \frac{(\theta_{32}+\sqrt{h^2+w^2})(h\sin\theta_{31} - w\cos\theta_{31})}{\sqrt{h^2+w^2}} \\ 1 & 0 & 1 \end{bmatrix}$$

Part(e): First, we'll find the kinematic singularities, where the jacobians lose rank. Simplifying, $det(J_1) = (\theta_{12} + w)^2$, which equals zero at $\theta_{12} = -w$. Similarly, $det(J_2) = (\theta_{22} + w)^2$, which equals zero at $\theta_{22} = -w$. Lastly, $det(J_3) = (\theta_{32} + \sqrt{h^2 + w^2})^2$, which equals zero at $\theta_{32} = -\sqrt{h^2 + w^2}$.

There are also actuator singularities, whenever the three prismatic joints all intersect at one point. This can happen in a variety of configurations.