ME 115(a): Solution to Homework #3

(Note: this problem set was corrected/amended, with the unnecessary problems removed. This solution represents the amended solution set)

Problem #1: (Problem 6(a,b,d) in Chapter 2 of MLS).

Part (a): Let Q and P be unit quaternions—i.e., $QQ^* = PP^* = 1$. The set of unit quaternions is a group if you can show that: (i) multiplication is commutative; (ii) the product of group elements yields a group element; (iii) the set contains an identity element; (iv) every group element has an inverse element, and the inverse is in the group.

(i) It is easy to show that quaternion multiplication is commutative.

(ii) The product of unit quaternions, QP, is a unit quaternion: $QP(QP)^* = QPP^*Q* = QQ^* = 1$.

(iii) The identity quaternion is: $e = (1, 0, 0, 0) = 1 + 0 \cdot i + 0 \cdot j + 0 \cdot k$.

(iv) The inverse of any unit quaternion Q is Q^*, which is also a unit quaternion (since $Q^*(Q^*)^* = Q^*Q = (QQ^*)^* = 1^* = 1$).

Part (b): If a unit quaternion, q, has real part q_R and pure part q_P, and $\vec{x} = [x_1\ x_2\ x_3]^T$ is represented as a pure quaternion $\tilde{x} = (0, x_1, x_2, x_3) = 0 + \vec{x}$, then:

$$\begin{aligned} \tilde{x}q^{-1} &= \vec{x} \cdot q_P \qquad (\leftarrow \text{real part}) \\ &+ q_R\vec{x} - (\vec{x} \times q_P \qquad (\leftarrow \text{pure part}) \end{aligned}$$

Similarly, the product $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q\tilde{x}q^{-1} &= q_R(\vec{x} \cdot q_P) - q_P \cdot (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{real part}) \\ &+ q_R(q_R\vec{x} - \vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{pure part}) \end{aligned}$$

The real part of $q\tilde{x}q^{-1}$ is:

$$(\vec{x} \cdot q_P)q_R - q_P \cdot [q_R\vec{x} - (\vec{x} \times q_P)] = q_R(\vec{x} \cdot q_P) - q_R(\vec{x} \cdot q_P) + q_P \cdot (\vec{x} \times q_P) = 0$$

Thus $q\tilde{x}q^{-1}$ is a pure quaternion when $\tilde{x}$ is.

The vector part of $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q_R(q_R\vec{x} - \vec{x} \times q_P) &+ (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \\ &= {q_R}^2\vec{x} - q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x}) - q_P \times (\vec{x} \times q_P) \\ &= q_R^2\vec{x} - 2q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P - [(q_P \cdot q_P)\vec{x} - (\vec{x} \cdot q_P)q_P] \\ &= [q_R^2 - (q_P \cdot q_P)]\vec{x} + 2[(\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x})] \end{aligned}$$

where we have used the triple cross product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Part (d):

(i) If $A_1, A_2 \in SO(3)$, then each of the 9 elements in the product matrix $A_1\ A_2$ requires 3 multiplications and 2 additions. Hence, the product $A_1\ A_2$ requires a total of 27 multiplications and 18 additions.

(ii) Let q_1 and q_2 be quaternions, with respective real and vector parts q_{1R}, q_{2R} and $\vec{q}_{1P}$, $\vec{q}_{2P}$. The real part of the quaternion product, $q_{1R}q_{2R} - \vec{q}_{1P} \cdot \vec{q}_{2P}$, requires 4 multiplications and 3 additions (where the subtraction is counted as an addition). The pure part, $\vec{q}_{3P} = q_{1R}\vec{q}_{2P} + q_{2R}\vec{q}_{1P} + \vec{q}_{1P} \times \vec{q}_{2P}$, can be evaluated in 12 multiplications and 9 additions. Thus, the quaternion product requires a total of 16 multiplications and 12 additions. It is therefore more efficient than the equivalent matrix multiplication.

(iii) The rotation of a vector by multiplication of a 3×3 rotation matrix times a 3×1 vector requires only 9 multiplications and 6 additions.

(iv) The number of multiplications and additions for the equivalent quaternion operation will depend upon the form which one uses for the quaternion vector rotation. Using the identity $1 = q_R^2 + q_P \cdot q_P$, it is possible to show that the vector part of $q\tilde{x}q^{-1}$ in part (b) above can be rearranged to the form:

$$\vec{x} + 2[q_P \times (q_P \times \vec{x}) + q_R(q_P \times \vec{x})]$$

Since $q_P \times \vec{x}$ need only be evaluated once, this takes only 18 multiplications and 12 additions. However, no matter what form one tries, the quaternion approach will always take more operations than the matrix/vector approach for vector rotation.

Problem # 3: (Problem 10(b,c) in Chapter 2 of the MLS text)

Part (b): If $\omega \in \mathbb{R}$, then let:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega J \qquad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that:

$$\hat{\omega}^2 = -\omega^2 I; \qquad \hat{\omega}^3 = \omega^3 J$$

Hence:

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \omega\theta J + \tfrac{(\omega\theta)^2}{2!}J^2 + \tfrac{(\omega\theta)^3}{3!}J^3 + \cdots \\ &= I + (\omega\theta)J - \tfrac{(\omega\theta)^2}{2!}I - \tfrac{(\omega\theta)^3}{3!}J + \cdots \\ &= (1 + \tfrac{(\omega\theta)^2}{2!} + \tfrac{(\omega\theta)}{4!} + \cdots)I + (\omega\theta - \tfrac{(\omega\theta)^3}{3!} + \cdots)J \\ &= \cos(\omega\theta)I + \sin(\omega\theta)J \end{aligned}$$

$$= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix}$$

Part (c): This can be shown by brute-force calculation. If $R \in SO(2)$ then:

$$\begin{aligned} R\hat{\omega}R^T &= \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}0 & -\omega \\ \omega & 0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix} \\ &= \omega\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta\end{bmatrix} \\ &= \omega\begin{bmatrix}0 & -(\cos^2\theta + \sin^2\theta) \\ (\sin^2\theta + \cos^2\theta) & 0\end{bmatrix}\omega\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix} = \hat{\omega} \end{aligned}$$

Problem # 5:

Part (a): Elements of $SU(2)$ have the form:

$$\begin{bmatrix}\mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^*\end{bmatrix} = \begin{bmatrix}(a+ib) & (c+id) \\ -(c-id) & (a-ib)\end{bmatrix}$$

where $zz^* + ww^* = a^2 + b^2 + c^2 + d^2 = 1$. Thus, the scalar elements a, b, c, and d are in one-to-one correspondence with the scalar elements of unit quaternions. That is, let a quaternion be represented by $q = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The correspondence is then:

$$\lambda_1 = a = Re(z) = \frac{z + z^*}{2} \tag{1}$$

$$\lambda_2 = b = Im(z) = \frac{i(z^* - z)}{2} \tag{2}$$

$$\lambda_3 = c = Re(w) = \frac{w + w^*}{2} \tag{3}$$

$$\lambda_4 = d = Im(w) = \frac{i(w^* - w)}{2} \tag{4}$$

The unit quaternion elements are in one-to-one correspondence with the Euler parameters of a rotation: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\cos\frac{\phi}{2}, s_x \sin\frac{\phi}{2}, s_y \sin\frac{\phi}{2}, s_z \sin\frac{\phi}{2})$. ϕ is the rotation about an axis represented by a unit vector $\vec{\omega} = [\omega_x\ \omega_y\ \omega_z]^T$. A 2×2 complex matrix which represents an arbitrary rotation as a function of the z-y-x Euler angles can be developed as the product of 2×2 complex matrices which represent rotations about the z, y, and x axes. A rotation about the x-axis of amount γ has the 2×2 representation (since $\lambda_1 = \cos\frac{\gamma}{2}$, $\lambda_2 = \sin\frac{\gamma}{2}$, $\lambda_3 = \lambda_4 = 0$):

$$\begin{bmatrix}(\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2}) & 0 \\ 0 & (\cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2})\end{bmatrix}\begin{bmatrix}e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}}\end{bmatrix}$$

Similarly, a rotation of amount ϕ about the y-axis can be represented as:

$$\begin{bmatrix}\cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{bmatrix}$$

while a rotation of amount ψ about the z-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}$$

Part (c):

$$\phi = 2\cos^{-1}(a) = 2\cos^{-1}(\frac{\mathbf{z}+\mathbf{z}^*}{2}) \tag{5}$$

$$\omega_x = \frac{b}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{z}-\mathbf{z}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{6}$$

$$\omega_y = \frac{c}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}+\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{7}$$

$$\omega_z = \frac{d}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}-\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{8}$$

Problem # 6:

Starting with Rodriguez' formula for a rotation matrix, and using half-angle identities, one can show that if $q = (a, b, c, d)$ is a unit quaternion, the formula for a rotation matrix in terms of unit quaternions is:

$$R = \begin{bmatrix} (a^2+b^2-c^2-d^2) & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & (a^2-b^2-c^2+d^2) \end{bmatrix}$$

To extract the quaternion parameters from the rotation matrix, note that:

$$r_{11}+r_{22}+r_{33} = 3a^2-b^2-c^2-d^2$$

Using the fact that $a^2+b^2+c^2+d^2 = 1$, then

$$r_{11}+r_{22}+r_{33} = 4a^1-1 \;\rightarrow\; a^2 = \frac{r_{11}+r_{22}+r_{33}+1}{4}$$

Assuming that $a \neq 0$, then

$$r_{21}-r_{12} = 2(bc+ad)-2(bc-ad) = 4ad \;\Rightarrow\; d = \frac{r_{21}-r_{12}}{4a}$$

Similary, one can show that

$$c = \frac{r_{13}-r_{31}}{4a} \qquad b = \frac{r_{32}-r_{23}}{4a}$$

If $a = 0$ (or practically, if a is very small), then one can choose a different factorization.