ME 115(a): Solution to Homework #3

Problem #1: (Prob. 10(b,c) in Chapt. 2 of MLS)

Part (b): If $\omega \in \mathbb{R}$, then let:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega J \qquad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that:

$$\hat{\omega}^2 = -\omega^2 I; \qquad \hat{\omega}^3 = \omega^3 J$$

Hence:

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \omega\theta J + \tfrac{(\omega\theta)^2}{2!}J^2 + \tfrac{(\omega\theta)^3}{3!}J^3 + \cdots \\ &= I + (\omega\theta)J - \tfrac{(\omega\theta)^2}{2!}I - \tfrac{(\omega\theta)^3}{3!}J + \cdots \\ &= (1 + \tfrac{(\omega\theta)^2}{2!} + \tfrac{(\omega\theta)^4}{4!} + \cdots)I + (\omega\theta - \tfrac{(\omega\theta)^3}{3!} + \cdots)J \\ &= \cos(\omega\theta)I + \sin(\omega\theta)J \\ &= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix} \end{aligned}$$

We can think of the map $exp : so(2) \to SO(3)$ as embedding a planar rotation into $SO(3)$, where we have chosen the axis of rotation to be perpendicular to the x-y plane:

$$\begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) & 0 \\ \sin(\omega\theta) & \cos(\omega\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly this map can not be surjective, since there are many rotations in $SO(3)$ which can not be expressed as the exponential of an element in $so(2)$. If we assume that the product $\omega\theta$ is nonzero except at $\theta = 0$ (i.e., assume that $\omega \neq 0$ and $\theta \in [-\pi, \pi]$), then the map is not injective since $exp(\omega\theta) = exp(\omega\theta + 2\pi)$.

Part (c): This can be shown by brute-force calculation. If $R \in SO(2)$ then:

$$\begin{aligned} R\hat{\omega}R^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \omega \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \\ &= \omega \begin{bmatrix} 0 & -(\cos^2\theta + \sin^2\theta) \\ (\sin^2\theta + \cos^2\theta) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega} \end{aligned}$$

Problem #2: (Prob. 11(a,b) in Chapt. 2 of MLS)

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of pure translation, or $\xi = (v, 0)$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi' = (h\hat{\xi}' h^{-1})^{\vee} \tag{1}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (1) gives:

$$\xi = (h\hat{\xi}' h^{-1})^{\vee} = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^{\vee} = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.

Problem #3: (Prob. 12 in Chapt. 2 of MLS

The given formula is very similar to Rodrigues' formula, but for $||\omega|| \neq 1$. Starting with Rodrigues' formula, we have:

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin(\theta) + \hat{\omega}^2(1 - \cos(\theta))$$

for $||\omega|| = 1$. For any $\omega \in \mathbb{R}^3$, we can express it as $\omega = ||\omega||\omega^*$ where $||\omega^*|| = 1$. Similarly, $\hat{\omega} = ||\omega||\hat{\omega}^*$. Then, employing Rodrigues' formula for the unit vector ω^*,

$$\begin{aligned} e^{\hat{\omega}\theta} &= e^{\hat{\omega}^*(||\omega||\theta)} \\ &= I + \hat{\omega}^*\sin(||\omega||\theta) + (\hat{\omega}^*)^2(1 - \cos(||\omega||\theta)) \\ &= I + \tfrac{\hat{\omega}}{||\omega||}\sin(||\omega||\theta) + \tfrac{\hat{\omega}^2}{||\omega^2||}(1 - \cos(||\omega||\theta)) \end{aligned}$$

Problem #4

Part (a): Elements of $SU(2)$ have the form:

$$\begin{bmatrix} \mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^* \end{bmatrix} = \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix}$$

where $zz^* + ww^* = a^2 + b^2 + c^2 + d^2 = 1$. To show that the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, K = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

form a basis for $SU(2)$, note that

$$
\begin{aligned}
aE + bI + cJ + dK &= a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\
&= \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^* \end{bmatrix}
\end{aligned}
$$

To show that the elements of $SU(2)$ are isomorphic to the unit quaternians, it is easily verified that the elements (E, I, J, K) satisfy the same relations as the basis elements of unit quaternians. I.E.:

$$
\begin{gathered}
aE = Ea \qquad aI = Ia \qquad aJ = Ja \qquad aK = Ka \qquad a \in \mathbb{R} \\
I \cdot I = J \cdot J = K \cdot K = I \cdot J \cdot K = -E \\
I \cdot J = -J \cdot I = K \qquad J \cdot K = -K \cdot J = I \qquad K \cdot I = -I \cdot K = J
\end{gathered}
$$

Thus, the scalar elements a, b, c, and d are in one-to-one correspondence with the scalar elements of unit quaternions. That is, let a quaternion be represented by $q = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The correspondence is then:

$$\lambda_1 = a = Re(z) = \frac{z + z^*}{2} \tag{2}$$

$$\lambda_2 = b = Im(z) = \frac{i(z^* - z)}{2} \tag{3}$$

$$\lambda_3 = c = Re(w) = \frac{w + w^*}{2} \tag{4}$$

$$\lambda_4 = d = Im(w) = \frac{i(w^* - w)}{2} \tag{5}$$

Part (b):

The unit quaternion elements are in one-to-one correspondence with the Euler parameters of a rotation: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\cos\frac{\phi}{2}, s_x \sin\frac{\phi}{2}, s_y \sin\frac{\phi}{2}, s_z \sin\frac{\phi}{2})$. ϕ is the rotation about an axis represented by a unit vector $\vec{s} = [s_x\ s_y\ s_z]^T$. A 2×2 complex matrix which represents an arbitrary rotation as a function of the z-y-x Euler angles can be developed as the product of 2×2 complex matrices which represent rotations about the z, y, and x axes. A rotation about the x-axis of amount γ has the 2×2 representation (since $\lambda_1 = \cos\frac{\gamma}{2}$, $\lambda_2 = \sin\frac{\gamma}{2}$, $\lambda_3 = \lambda_4 = 0$):

$$
\begin{bmatrix} (\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2}) & 0 \\ 0 & (\cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2}) \end{bmatrix} = \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix}
$$

Similarly, a rotation of amount ϕ about the y-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$

while a rotation of amount ψ about the z-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}$$

Therefore, the special unitary representation of a rotation in terms of z-y-x Euler Angles can be computed as:

$$\begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix} \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix}$$

Part (c):

$$\phi = 2\cos^{-1}(a) \tag{6}$$

$$s_x = \frac{b}{\sqrt{b^2 + c^2 + d^2}} \tag{7}$$

$$s_y = \frac{c}{\sqrt{b^2 + c^2 + d^2}} \tag{8}$$

$$s_z = \frac{d}{\sqrt{b^2 + c^2 + d^2}} \tag{9}$$