ME 115(a): Solution to Homework #3

Problem #1: (Problem 6(b,d) in Chapter 2 of MLS).

Part (b): If a unit quaternion, q, has real part q_R and pure part q_P, and $\vec{x} = [x_1\ x_2\ x_3]^T$ is represented as a pure quaternion $\tilde{x} = (0, x_1, x_2, x_3) = 0 + \vec{x}$, then:

$$\begin{aligned} \tilde{x}q^{-1} &= \vec{x} \cdot q_P \qquad (\leftarrow \text{real part}) \\ &+ \quad q_R\vec{x} - (\vec{x} \times q_P \qquad (\leftarrow \text{pure part}) \end{aligned}$$

Similarly, the product $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q\tilde{x}q^{-1} &= q_R(\vec{x} \cdot q_P) - q_P \cdot (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{real part}) \\ &+ \quad q_R(q_R\vec{x} - \vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \qquad (\leftarrow \text{pure part}) \end{aligned}$$

The real part of $q\tilde{x}q^{-1}$ is:

$$(\vec{x} \cdot q_P)q_R - q_P \cdot [q_R\vec{x} - (\vec{x} \times q_P)] = q_R(\vec{x} \cdot q_P) - q_R(\vec{x} \cdot q_P) + q_P \cdot (\vec{x} \times q_P) = 0$$

Thus $q\tilde{x}q^{-1}$ is a pure quaternion when $\tilde{x}$ is.

The vector part of $q\tilde{x}q^{-1}$ is:

$$\begin{aligned} q_R(q_R\vec{x} - \vec{x} \times q_P) &+ (\vec{x} \cdot q_P)q_P + q_P \times (q_R\vec{x} - \vec{x} \times q_P) \\ &= {q_R}^2\vec{x} - q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x}) - q_P \times (\vec{x} \times q_P) \\ &= q_R^2\vec{x} - 2q_R(\vec{x} \times q_P) + (\vec{x} \cdot q_P)q_P - [(q_P \cdot q_P)\vec{x} - (\vec{x} \cdot q_P)q_P] \\ &= [q_R^2 - (q_P \cdot q_P)]\vec{x} + 2[(\vec{x} \cdot q_P)q_P + q_R(q_P \times \vec{x})] \end{aligned}$$

where we have used the triple cross product identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Part (d):

(i) If $A_1, A_2 \in SO(3)$, then each of the 9 elements in the product matrix $A_1\ A_2$ requires 3 multiplications and 2 additions. Hence, the product $A_1\ A_2$ requires a total of 27 multiplications and 18 additions.

(ii) Let q_1 and q_2 be quaternions, with respective real and vector parts q_{1R}, q_{2R} and $\vec{q}_{1P}$, $\vec{q}_{2P}$. The real part of the quaternion product, $q_{1R}q_{2R} - \vec{q}_{1P} \cdot \vec{q}_{2P}$, requires 4 multiplications and 3 additions (where the subtraction is counted as an addition). The pure part, $\vec{q}_{3P} = q_{1R}\vec{q}_{2P} + q_{2R}\vec{q}_{1P} + \vec{q}_{1P} \times \vec{q}_{2P}$, can be evaluated in 12 multiplications and 9 additions. Thus, the quaternion product requires a total of 16 multiplications and 12 additions. It is therefore more efficient than the equivalent matrix multiplication.

(iii) The rotation of a vector by multiplication of a 3×3 rotation matrix times a 3×1 vector requires only 9 multiplications and 6 additions.

(iv) The number of multiplications and additions for the equivalent quaternion operation will depend upon the form which one uses for the quaternion vector rotation. Using the identity $1 = q_R^2 + q_P \cdot q_P$, it is possible to show that the vector part of $q\tilde{x}q^{-1}$ in part (b) above can be rearranged to the form:

$$\vec{x} + 2[q_P \times (q_P \times \vec{x}) + q_R(q_P \times \vec{x})]$$

Since $q_P \times \vec{x}$ need only be evaluated once, this takes only 18 multiplications and 12 additions. However, no matter what form one tries, the quaternion approach will always take more operations than the matrix/vector approach for vector rotation.

Problem #2: (Prob. 10(b,c) in Chapt. 2 of MLS)

Part (b): If $\omega \in \mathbb{R}$, then let:

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} = \omega J \qquad \text{where } J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that:

$$\hat{\omega}^2 = -\omega^2 I; \qquad \hat{\omega}^3 = \omega^3 J$$

Hence:

$$\begin{aligned} e^{\hat{\omega}\theta} &= I + \omega\theta J + \tfrac{(\omega\theta)^2}{2!}J^2 + \tfrac{(\omega\theta)^3}{3!}J^3 + \cdots \\ &= I + (\omega\theta)J - \tfrac{(\omega\theta)^2}{2!}I - \tfrac{(\omega\theta)^3}{3!}J + \cdots \\ &= (1 + \tfrac{(\omega\theta)^2}{2!} + \tfrac{(\omega\theta)^4}{4!} + \cdots)I + (\omega\theta - \tfrac{(\omega\theta)^3}{3!} + \cdots)J \\ &= \cos(\omega\theta)I + \sin(\omega\theta)J \\ \\ &= \begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) \\ \sin(\omega\theta) & \cos(\omega\theta) \end{bmatrix} \end{aligned}$$

We can think of the map $exp : so(2) \to SO(3)$ as embedding a planar rotation into $SO(3)$, where we have chosen the axis of rotation to be perpendicular to the x-y plane:

$$\begin{bmatrix} \cos(\omega\theta) & -\sin(\omega\theta) & 0 \\ \sin(\omega\theta) & \cos(\omega\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Clearly this map can not be surjective, since there are many rotations in $SO(3)$ which can not be expressed as the exponential of an element in $so(2)$. If we assume that the product $\omega\theta$ is nonzero except at $\theta = 0$ (i.e., assume that $\omega \neq 0$ and $\theta \in [-\pi, \pi]$), then the map is not injective since $exp(\omega\theta) = exp(\omega\theta + 2\pi)$.

Part (c): This can be shown by brute-force calculation. If $R \in SO(2)$ then:

$$\begin{aligned} R\hat{\omega}R^T &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \omega \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} \\ &= \omega \begin{bmatrix} 0 & -(\cos^2\theta + \sin^2\theta) \\ (\sin^2\theta + \cos^2\theta) & 0 \end{bmatrix} = \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{\omega} \end{aligned}$$

Problem #3: (Prob. 11(a,b) in Chapt. 2 of MLS)

Part (a): Recall that the matrix exponential of a twist, $\hat{\xi}$, is:

$$e^{\phi\hat{\xi}} = I + \frac{\phi}{1!}\hat{\xi} + \frac{\phi^2}{2!}\hat{\xi}^2 + \frac{\phi^3}{3!}\hat{\xi}^3 + \cdots$$

First, let's consider the case of $\xi = (v, \omega)$, with $\omega = 0$. If:

$$\hat{\xi} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

then $\hat{\xi}^2 = 0$. Thus

$$e^{\phi\hat{\xi}} = \begin{bmatrix} 1 & 0 & \phi v_x \\ 0 & 1 & \phi v_y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I & \vec{v}\phi \\ \vec{0}^t & 1 \end{bmatrix}$$

To compute the exponential for the more general case in which $\omega \neq 0$, let us assume that $||\omega|| = 1$. In this case, note that $\hat{\omega}^2 = -I$, where I is the 2×2 identity matrix. It is easiest if we choose a different coordinate system in which to perform the calculations. Let

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & \vec{v} \\ \vec{0}^T & 0 \end{bmatrix}$$

Let

$$g = \begin{bmatrix} I & \hat{\omega}\vec{v} \\ \vec{0}^T & 1 \end{bmatrix}$$

Let is define a new twist, $\hat{\xi}'$:

$$\begin{aligned} \hat{\xi}' &= g^{-1}\hat{\xi}g \\ &= \begin{bmatrix} I & -\hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & \vec{v} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & \hat{\omega}\vec{v} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \hat{\omega} & (\hat{\omega}^2\vec{v} + \vec{v}) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

where we made use of the identity $\hat{\omega}^2 = -I$. That is, we have chosen a coordinate system in which $\hat{\xi}'$ corresponds to a pure rotation. Thus,

$$e^{\phi\hat{\xi}'} = \begin{bmatrix} e^{\phi\hat{\omega}} & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Eq. (2.35) on page 42 of the MLS text:

$$e^{\phi\hat{\xi}} = g e^{\phi\hat{\xi}'} g^{-1} = \begin{bmatrix} e^{\phi\hat{\omega}} & (I - e^{\phi\hat{\omega}})\hat{\omega}\vec{v}\phi \\ 0 & 1 \end{bmatrix}$$

which is clearly an element of $SE(2)$.

Part(b): It is easy to see from part (a) that the twist $\xi = (v_x, v_y, 0)^T$ maps directly to the planar translation (v_x, v_y).

The twist corresponding to pure rotation about a point $\vec{q} = (q_x, q_y)$ can be thought of as the Ad-transformation of a twist, $\xi' = (0, 0, \omega)$, which is pure rotation, by a transformation, g, which is pure translation by $\vec{q}$:

$$\xi = \mathrm{Ad}_h \xi' = (h\hat{\xi}' h^{-1})^\vee \tag{1}$$

where

$$h = \begin{bmatrix} I & \vec{q} \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \hat{xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ \vec{0}^T & 0 \end{bmatrix}.$$

Expanding Eq. (1) gives:

$$\xi = (h\hat{\xi}' h^{-1})^\vee = \begin{bmatrix} \hat{\omega} & -\hat{\omega}\vec{q} \\ \vec{0}^T & 0 \end{bmatrix}^\vee = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$$

assuming $\omega = 1$.

Problem #4:

Part (a): From the handout on screw displacements and Rodriguez's spatial displacement formula:

$$\vec{\omega} \tan\frac{\phi}{2} = \frac{[(Q_1 - Q_0) - (R_1 - R_0)] \times [(P_1 - P_0) - (R_1 - R_0)]}{[(Q_1 - Q_0) - (R_1 - R_0)] \cdot [(P_1 + P_0) - (R_1 + R_0)]}$$

Substituting in the relevant data:

$$\tan\frac{\phi}{2}\vec{\omega} = \begin{bmatrix} 0.3333 \\ 0.3333 \\ 0.3333 \end{bmatrix}$$

which yields $\phi = 60^o$ and $\vec{\omega} = [0.5773,\ 0.5773,\ 0.5773]^T = \frac{1}{\sqrt{3}}[1,\ 1,\ 1]^T$.

Also from the handout we have:

$$\vec{\rho}_\perp = \frac{1}{2}\left[\frac{\vec{\omega} \times (P_1 - P_0)}{\tan\frac{\phi}{2}} - [\vec{\omega} \cdot (P_0 + P_1)]\vec{\omega} + P_0 + P_1\right]$$

Substituting in yields: $\vec{\rho}_\perp \simeq 0$. $d^{||}$ can be computed as:

$$d^{||} = \overline{s} \cdot (P_1 - P_0) = 3.464$$

Part (b): We know that the rotation matrix A can be found as $A = I + \sin\phi\,\hat{\omega} + (1 - \cos\phi)\,\hat{\omega}^2$:

$$\begin{bmatrix} 0.66667 & -0.3333 & 0.66667 \\ 0.66667 & 0.66667 & -0.3333 \\ -0.3333 & 0.66667 & 0.66667 \end{bmatrix}$$

Similarly, the displacement vector is $\vec{d} = d^{||}\vec{\omega} + (I - A)\vec{\rho}_{\perp}$:

$$\vec{d} = [2,\ 2,\ 2]^T$$

Problem #5

Part (a): Elements of $SU(2)$ have the form:

$$\begin{bmatrix} \mathbf{z} & \mathbf{w} \\ -\mathbf{w}^* & \mathbf{z}^* \end{bmatrix} = \begin{bmatrix} (a+ib) & (c+id) \\ -(c-id) & (a-ib) \end{bmatrix}$$

where $zz^* + ww^* = a^2 + b^2 + c^2 + d^2 = 1$. Thus, the scalar elements a, b, c, and d are in one-to-one correspondence with the scalar elements of unit quaternions. That is, let a quaternion be represented by $q = \lambda_1 + \lambda_2 i + \lambda_3 j + \lambda_4 k = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The correspondence is then:

$$\lambda_1 = a = Re(z) = \frac{z + z^*}{2} \tag{2}$$

$$\lambda_2 = b = Im(z) = \frac{i(z^* - z)}{2} \tag{3}$$

$$\lambda_3 = c = Re(w) = \frac{w + w^*}{2} \tag{4}$$

$$\lambda_4 = d = Im(w) = \frac{i(w^* - w)}{2} \tag{5}$$

The unit quaternion elements are in one-to-one correspondence with the Euler parameters of a rotation: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\cos\frac{\phi}{2}, s_x \sin\frac{\phi}{2}, s_y \sin\frac{\phi}{2}, s_z \sin\frac{\phi}{2})$. ϕ is the rotation about an axis represented by a unit vector $\vec{\omega} = [\omega_x\ \omega_y\ \omega_z]^T$. A 2×2 complex matrix which represents an arbitrary rotation as a function of the z-y-x Euler angles can be developed as the product of 2×2 complex matrices which represent rotations about the z, y, and x axes. A rotation about the x-axis of amount γ has the 2×2 representation (since $\lambda_1 = \cos\frac{\gamma}{2}$, $\lambda_2 = \sin\frac{\gamma}{2}$, $\lambda_3 = \lambda_4 = 0$):

$$\begin{bmatrix} (\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2}) & 0 \\ 0 & (\cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2}) \end{bmatrix} = \begin{bmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{bmatrix}$$

Similarly, a rotation of amount ϕ about the y-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{bmatrix}$$

while a rotation of amount ψ about the z-axis can be represented as:

$$\begin{bmatrix} \cos\frac{\psi}{2} & i\sin\frac{\psi}{2} \\ i\sin\frac{\psi}{2} & \cos\frac{\psi}{2} \end{bmatrix}$$

Part (c):

$$\phi = 2\cos^{-1}(a) = 2\cos^{-1}(\frac{\mathbf{z}+\mathbf{z}^*}{2}) \tag{6}$$

$$\omega_x = \frac{b}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{z}-\mathbf{z}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{7}$$

$$\omega_y = \frac{c}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}+\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{8}$$

$$\omega_z = \frac{d}{\sqrt{b^2+c^2+d^2}} = \frac{(\mathbf{w}-\mathbf{w}^*)/2}{\sqrt{(\frac{\mathbf{z}-\mathbf{z}^*}{2})^2+\mathbf{w}\mathbf{w}^*}} \tag{9}$$