ME 115B Solution to Homework #3

Problem #1:

The manipulability μ_3 is defined to be

$$\mu_3 = \det(J(\theta)).$$

We can decompose J using singular decomposition. Thus

$$J = U\Sigma V^T.$$

Therefore,

$$\mu_3 = \det(U)\det(\Sigma)\det(V).$$

Since U and V are orthogonal,

$$\mu_3 = \det(\Sigma) = \Pi_{i=1}^{6}\sigma_i(\theta)$$

by the structure of Σ.

Problem #2:

We hope to find singularities of ZYZ Euler angles. This will happen when the Jacobian of the forward kinematics loses rank. The twists for the "virtual" joints are given by:

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Using the formula for the spatial Jacobian, we get

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\sin(\alpha) & \cos(\alpha)\sin(\beta) \\ 0 & \cos(\alpha) & \sin(\alpha)\sin(\beta) \\ 1 & 0 & \cos(\beta) \end{bmatrix}$$

J loses rank where $\det(J^T J) = 0$, or

$$\det \begin{bmatrix} 1 & 0 & \cos(\beta) \\ 0 & 1 & 0 \\ \cos(\beta) & 0 & 1 \end{bmatrix} = 0$$

or $\cos^2(\beta) = 1$.Thus, the ZYZ Euler angle parameterization has a singularity at $\beta = 0°, 180°$.

Problem #3:

a) Using Gruebler's formula for a planar linkage (as stated in the book), we have

$$\text{F} = 3\text{N} - \sum(3 - f_i) = 3(N - g) + \sum f_i$$

In our case, N = 7, g =9, and all of the $f_i = 1$,so F = 3(7-9) +9 = 3.

b) The structure equations are of the form:

$$g_{st} = e^{\xi_{11}\theta_{11}} e^{\xi_{12}\theta_{12}} e^{\xi_{13}\theta_{13}} g_{st}(0) = e^{\xi_{21}\theta_{21}} e^{\xi_{22}\theta_{22}} e^{\xi_{23}\theta_{23}} g_{st}(0) = e^{\xi_{31}\theta_{31}} e^{\xi_{32}\theta_{32}} e^{\xi_{33}\theta_{33}} g_{st}(0)$$

Again using planar notation, and placing the stationary and tool frames where they appear in the figure in the book, we have:

$$\xi_{11} = \begin{bmatrix} \frac{h}{2} \\ 0 \\ 1 \end{bmatrix} ; \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \xi_{13} = \begin{bmatrix} \frac{h}{2} \\ -w \\ 1 \end{bmatrix}$$

$$\xi_{11} = \begin{bmatrix} -\frac{h}{2} \\ 0 \\ 1 \end{bmatrix} ; \xi_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \xi_{13} = \begin{bmatrix} -\frac{h}{2} \\ -w \\ 1 \end{bmatrix}$$

$$\xi_{11} = \begin{bmatrix} -\frac{h}{2} \\ 0 \\ 1 \end{bmatrix} ; \xi_{12} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} ; \xi_{13} = \begin{bmatrix} \frac{h}{2} \\ -w \\ 1 \end{bmatrix}$$

where $\alpha = \tan^{-1}\left(\frac{h}{w}\right)$. Also note that the zero configuration of the tool frame is given by

$$g_{st}(0) = \begin{bmatrix} 1 & 0 & w \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the structure equations are just the simplification of the above expressions.

c) The structure equations can be solved explicitly for d_1, d_2,and d_3, or they can be found geometrically. From the geometry, we can find the position of the endpoints of each link in terms of $g_{st} = ([x, y], R_\phi)$, which leads to an expression for the length of that link.

d_1 connects the points $(0, \frac{h}{2})$ and $(x - \frac{h}{2}\sin\phi, y + \frac{h}{2}\cos\phi)$, which leads to the equation

$$d_1^2 = (x - \tfrac{h}{2}\sin\phi)^2 + (y + \tfrac{h}{2}\cos\phi - \tfrac{h}{2})^2$$

or,

$$d_1 = ((x - \tfrac{h}{2}\sin\phi)^2 + (y + \tfrac{h}{2}\cos\phi - \tfrac{h}{2})^2)^{\frac{1}{2}}$$

Similarly, for d_2 and d_3 gives

$$d_2 = ((x + \tfrac{h}{2}\sin\phi)^2 + (y - \tfrac{h}{2}\cos\phi + \tfrac{h}{2})^2)^{\frac{1}{2}}$$

$$d_3 = ((x - \tfrac{h}{2}\sin\phi)^2 + (y + \tfrac{h}{2}\cos\phi + \tfrac{h}{2})^2)^{\frac{1}{2}}$$

Problem #4:

a)
The grasp map for this problem is defined as:

G $= \begin{bmatrix} Ad^T_{g_{oc1^{-1}}} B_{c1} & Ad^T_{g_{oc2^{-1}}} B_{c2} & Ad^T_{g_{oc2^{-1}}} B_{c3} \end{bmatrix}$, where $Ad^T_{g_{oci^{-1}}} = \begin{bmatrix} R_{oc_i} & 0 \\ p^{\wedge}_{oci} R_{oc_i} & R_{oc_i} \end{bmatrix}, B_{ci} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Choosing a frame in the middle of the triangle:

$$Ad^T_{g_{oc1^{-1}}} B_{c1} = \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ a & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, Ad^T_{g_{oc2^{-1}}} B_{c2} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ a & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, Ad^T_{g_{oc3^{-1}}} B_{c3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where a is the distance from the frame to the point of contacts.

$$G = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & \frac{-1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The grasp map has a rank of 2 and cannot span R^3. The grasp is not force closure.

b) For this grasp, the transformations are given as:

$$Ad^T_{g_{oc1^{-1}}} = \begin{bmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ a & 0 & 1 \end{bmatrix}, Ad^T_{g_{oc2^{-1}}} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ a & b & 1 \end{bmatrix}, Ad^T_{g_{oc3^{-1}}} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ a & -b & 1 \end{bmatrix}, Ad^T_{g_{oc4^{-1}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix},$$

where b is half the distance between the two contacts located on that one side of the triangle.

$$G = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & \frac{-1}{2} & \frac{-1}{2} & 1 \\ 0 & b & -b & 0 \end{bmatrix}$$

G is full rank so the grasp is force closure.